2008

Inverse scattering problems for first-order systems

Jaemin Shin
Iowa State University

Inverse scattering problems for first-order systems

by

Jaemin Shin

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
Paul Sacks, Major Professor
Marshall Luban
Michael Smiley
Sung Yell Song
Zhijun Wu

Iowa State University

Ames, Iowa

2008

To Jung Hwa

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

In this thesis, the Zakharov-Shabat scattering problem and several types of Landau-Lifschitz scattering problems are considered. The inverse scattering problem is that of seeking one or more coefficients in a system of differential equation from the scattering data which, generally, consists of a reflection coefficient and bound state data.

We assume mainly that the coefficients to be determined are of half line support, i.e. they are equal to zero on a half line. Such cases can arise on natural physical grounds, and they can be very good approximations in case of reasonably rapid decay.

Assuming that the coefficients have half line support, we present the uniqueness of the inverse scattering problems as well as develop several efficient numerical algorithms to reconstruct the coefficients via a time domain approach. Also, a relation of the Zakharov-Shabat scattering problem and the Landau-Lifschitz scattering problem is investigated. Some exact theory for the inverse scattering problem with no support restriction is developed by means of corresponding half line support problems.

CHAPTER 1. Introduction

Consider the following first-order system of equations,

$$\frac{\partial \Psi(x,\zeta)}{\partial x} = \Omega(x,\zeta)\Psi(x,\zeta), \quad x \in \mathbb{R}, \quad \zeta \in \mathbb{C}, \tag{1.1}$$

where $\Omega(x,\zeta)$ is a 2×2 matrix that converges to $\Omega_{\pm\infty}(\zeta)$ as $x \to \pm\infty$. Several equations which arise in many areas of physics and engineering are included in (1.1). For example, (1.1) represents the Schrödinger equation, a wave equation in an inhomogeneous elastic medium, the Zakharov-Shabat scattering problem (ZSSP), the Landau-Lifschitz scattering problem (LLSP) and the anisotropic model of Landau-Lifschitz scattering problem (ALLSP) with specific forms of $\Omega(x,\zeta)$ (refer to Table 1.1).

Table 1.1 Scattering problems with $\Omega(x,\zeta)$

scattering problem	$\Omega(x,\zeta)$
Schrödinger eqn.	$\begin{pmatrix} 0 & -\zeta^2 + u(x) \\ 1 & 0 \end{pmatrix}$
Wave eqn. in elastic medium	$i\zeta \begin{pmatrix} 0 & \xi^{-1}(x) \\ \xi(x) & 0 \end{pmatrix}$
ZSSP	$i\zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & s^*(x) \\ -s(x) & 0 \end{pmatrix}$
Generalized ZSSP	$i\zeta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & s^*(x) \\ r(x) & 0 \end{pmatrix}$
LLSP	$i\zeta \begin{pmatrix} q_3(x) & q^*(x) \\ q(x) & -q_3(x) \end{pmatrix}, \ \lvert q(x)\rvert^2 + q_3(x)^2 = 1$
ALLSP	$i\zeta \begin{pmatrix} q_3(x) & q^*(x) \\ q(x) & -q_3(x) \end{pmatrix} - \beta \begin{pmatrix} 0 & q^*(x) \\ -q(x) & 0 \end{pmatrix}$

In this thesis, we study the relation of the coefficient $\Omega(x,\zeta)$ and the asymptotic behavior of $\Psi(\cdot,\zeta)$. In Section 1.1, we define the asymptotic behavior of the solutions to (1.1) systematically as scattering data. To extract the scattering data is said to be the direct scattering problem.

Conversely, the inverse scattering problem is to seek the coefficient $\Omega(x,\zeta)$ from the given scattering data.

One motivation of the inverse scattering problem is the so-called Inverse Scattering Transformation which will be briefly discussed in Section 1.2.

1.1 Scattering data

Suppose that $\Omega(x,\zeta)$ is smooth enough and converges to $i\zeta\Omega_{\pm\infty}$ sufficiently rapidly for nonsingular constant matrices $\Omega_{\pm\infty}$ as $x\to\pm\infty$ respectively. Then one can define the matrix Jost solutions $J^+=[\mu\ \overline{\mu}]$ and $J^-=[\nu\ \overline{\nu}]$ to (1.1) such that

$$J^{\pm}(x,\zeta)\to e^{i\zeta\Omega_{\pm\infty}x}\quad \text{as } x\to\pm\infty. \tag{1.2}$$

Due to the nonsingularity of $\Omega_{+\infty}$, the vector Jost solutions $\mu(\cdot,\zeta)$ and $\overline{\mu}(\cdot,\zeta)$ are linearly independent. Similarly, $\nu(\cdot,\zeta)$ and $\overline{\nu}(\cdot,\zeta)$ are linearly independent. Note that $\overline{\mu},\overline{\nu}$ are not the complex conjugate of μ,ν. We use $*$ for the complex conjugate and $\dagger$ for the Hermitian conjugate of a matrix.

$J^{\pm}(x,\zeta)$ satisfy the following Volterra integral equations,

$$J^+(x,\zeta) = e^{i\zeta\Omega_{+\infty}x}-\int_x^{\infty}e^{i\zeta\Omega_{+\infty}(x-y)}(\Omega(y,\zeta)-i\zeta\Omega_{+\infty})J^+(y,\zeta)dy, \tag{1.3}$$

$$J^-(x,\zeta) = e^{i\zeta\Omega_{-\infty}x}+\int_{-\infty}^{x}e^{i\zeta\Omega_{-\infty}(x-y)}(\Omega(y,\zeta)-i\zeta\Omega_{-\infty})J^-(y,\zeta)dy. \tag{1.4}$$

Under suitable hypotheses they will have Neumann expansions,

$$J^{\pm}(x,\zeta)=\sum_{n=0}^{\infty}J_n^{\pm}(x,\zeta),\quad J_0^{\pm}(x,\zeta)=e^{i\zeta\Omega_{\pm\infty}x}, \tag{1.5}$$

where

$$\begin{aligned} J_{n+1}^+(x,\zeta) &= -\int_x^{\infty}e^{i\zeta\Omega_{+\infty}(x-y)}(\Omega(y,\zeta)-i\zeta\Omega_{+\infty})J_n^+(y,\zeta)dy,\\ J_{n+1}^-(x,\zeta) &= \int_{-\infty}^{x}e^{i\zeta\Omega_{-\infty}(x-y)}(\Omega(y,\zeta)-i\zeta\Omega_{-\infty})J_n^-(y,\zeta)dy. \end{aligned}$$

Suppose that for some functions $V_{\pm}(x)$,

$$|\Omega(x,\zeta)-i\zeta\Omega_{\pm\infty}|\le(|\zeta|^{\alpha}+|\beta|)V_{\pm}(x).$$

Here, constants $\alpha \geq 0$ and $\beta \in \mathbb{R}$ are related to the structure of $\Omega(x,\zeta)$[1], and $|\Omega(x,\zeta)|$ denotes the matrix L^2 norm, i.e. for fixed x, ζ

$$|\Omega(x,\zeta)| = \sqrt{\lambda_{\max}(\Omega^\dagger(x,\zeta)\Omega(x,\zeta))},$$

where $\lambda_{\max}$ means the largest eigenvalue.

Let

$$\begin{aligned} M_+(x,\zeta) &= \int_x^\infty e^{2|\zeta||\Omega_{+\infty}||y|} V_+(y)dy, \\ M_-(x,\zeta) &= \int_{-\infty}^x e^{2|\zeta||\Omega_{-\infty}||y|} V_-(y)dy. \end{aligned}$$

Then, it is not difficult to check that

$$\begin{aligned} |J_n^+(x,\zeta)| &\leq e^{|\zeta||\Omega_{+\infty}||x|} \frac{(|\zeta|^\alpha + |\beta|)^n M_+^n(x,\zeta)}{n!}, \\ |J_n^-(x,\zeta)| &\leq e^{|\zeta||\Omega_{-\infty}||x|} \frac{(|\zeta|^\alpha + |\beta|)^n M_-^n(x,\zeta)}{n!}. \end{aligned}$$

Assume that $\Omega(x,\zeta) \to i\zeta\Omega_{\pm\infty}$ sufficiently fast as $x \to \pm\infty$, for example, for $\varepsilon > 0$

$$V_\pm(x) \leq e^{-|x|^{1+\varepsilon}}, \quad \text{as } x \to \pm\infty. \tag{1.6}$$

Then the Neumann series (1.5) converge absolutely, thus unique Jost solutions $J^\pm(x,\zeta)$ exist and $J^\pm(x,\cdot)$ are entire functions in ζ-plane.

Since $\{\mu, \overline{\mu}\}$ and $\{\nu, \overline{\nu}\}$ are linearly independent, there is a matrix $\mathcal{T}(\zeta)$ such that

$$J^+(x,\zeta) = J^-(x,\zeta)\mathcal{T}(\zeta), \quad \mathcal{T}(\zeta) = \begin{pmatrix} a(\zeta) & \overline{b}(\zeta) \\ b(\zeta) & \overline{a}(\zeta) \end{pmatrix}. \tag{1.7}$$

Note that each component of $\mathcal{T}$ is an entire function assuming (1.6).

Assuming $\operatorname{trace}(\Omega(x,\zeta)) = 0$ and $\lambda(\Omega_{+\infty}) = \lambda(\Omega_{-\infty}) = \{-1, 1\}$[2], Liouville's formula yields

$$a(\zeta)\overline{a}(\zeta) - b(\zeta)\overline{b}(\zeta) = 1. \tag{1.8}$$

Furthermore, if $\Omega(x,\zeta)$ has a symmetry such that

$$\Omega^*(x,\zeta) = \sigma_y \Omega(x,\zeta^*)\sigma_y, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1.9}$$

[1]For example, $\alpha = 0$ is related to the ZSSP, $\alpha = 1$ is for the LLSP and ALLSP. β is the parameter appearing in the ALLSP. See Table 1.1 and Table 1.2 for the definitions of scattering problems.

[2]In case of $\Omega_{\pm\infty}(\zeta) \neq i\zeta\Omega_{\pm\infty}$, this condition may replace to $\lambda(\Omega_\infty(\zeta)) = \{-i\zeta, i\zeta\}$.

then

$$\overline{a}(\zeta) = a^*(\zeta^*), \quad \overline{b}(\zeta) = -b^*(\zeta^*). \tag{1.10}$$

Indeed, for any solution $\Psi(x,\zeta)$ to (1.1), $\sigma_y \Psi^*(x,\zeta)\sigma_y$ solves (1.1) with the coefficient $\Omega(x,\zeta^*)$. Thus

$$\sigma_y \mathcal{T}^*(\zeta)\sigma_y = \mathcal{T}(\zeta^*), \tag{1.11}$$

which gives (1.10). Note that the ZSSP, the LLSP, and the ALLSP satisfy the condition (1.9).

Assuming (1.10), we rewrite (1.7) as

$$\mu(x,\zeta) = a(\zeta)\nu(x,\zeta) + b(\zeta)\overline{\nu}(x,\zeta), \tag{1.12}$$

$$\overline{\nu}(x,\zeta) = b^*(\zeta^*)\mu(x,\zeta) + a(\zeta)\overline{\mu}(x,\zeta). \tag{1.13}$$

The left and right reflection coefficients and transmission coefficient are defined on the real line by

$$L(\zeta) = \frac{b(\zeta)}{a(\zeta)}, \quad R(\zeta) = \frac{b^*(\zeta)}{a(\zeta)}, \quad T(\zeta) = \frac{1}{a(\zeta)}, \quad \zeta \in \mathbb{R}. \tag{1.14}$$

In general, $a(\zeta)$ may have zeros on $\mathbb{R}$, although there exists $\Omega(x,\zeta)$ such that the corresponding $a(\zeta)$ never vanishes on the real line. Thus we make a technical assumption,

$$|a(\zeta)| > 0, \quad \text{for all } \zeta \in \mathbb{R}. \tag{1.15}$$

We have $a(\zeta) = 1$ for $\Omega(x,\zeta)$ which is constant with respect to x, say $\Omega(x,\zeta) = i\zeta\Omega_{+\infty}$, so the condition (1.15) is satisfied. Consider $a(\zeta;\Omega)$ as a function of Ω for ζ fixed in $\mathbb{R}$. Then one can show that $a(\zeta,\cdot)$ is continuous with respect Ω in a suitable class. Therefore (1.15) is satisfied generically ([20]).

A bound state is a square integrable vector solution to (1.1) which occurs at a certain $\zeta_0 \in \mathbb{C}$. For simplicity of current discussion, we assume that $\Omega_{\pm\infty} = \Lambda$,

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.16}$$

which is the most common case.

Let $\psi(x,\zeta_0)$ be a bound state for $\zeta=\zeta_0$. Since $\{\mu(x,\zeta_0),\overline{\mu}(x,\zeta_0)\}$ and $\{\nu(x,\zeta_0),\overline{\nu}(x,\zeta_0)\}$ are fundamental sets, there are constants c_1,c_2,d_1, and d_2 such that

$$\begin{aligned}\psi(x,\zeta_0) &= c_1\mu(x,\zeta_0)+c_2\overline{\mu}(x,\zeta_0) && (1.17)\\ &= d_1\nu(x,\zeta_0)+d_2\overline{\nu}(x,\zeta_0). && (1.18)\end{aligned}$$

Obviously $\mathrm{Im}\zeta_0\neq 0$ from the asymptotic behavior of the Jost solutions. Moreover, we have $c_2=d_1=0$ assuming $\mathrm{Im}\zeta_0>0$, thus there is a constant $\gamma_{r,0}$ such that

$$\mu(x,\zeta_0)=\gamma_{r,0}\overline{\nu}(x,\zeta_0).$$

With (1.12), $a(\zeta_0)=0$, and

$$\gamma_{r,0}=b(\zeta_0). \tag{1.19}$$

Similarly, one can define $\gamma_{l,0}$ as

$$\overline{\nu}(x,\zeta_0)=\gamma_{l,0}\mu(x,\zeta_0).$$

(1.13) and (1.8) imply

$$\gamma_{l,0}=b^*(\zeta_0^*)=1/\gamma_{r,0}.$$

Note that $\gamma_{l,r;0}$ are said to be dependency constants. Conversely, it is not difficult to show that there exist a bound state $\psi(x,\zeta_0)$ and dependency constants for every zero ζ_0 of $a(\zeta)$ in $\mathbb{C}^+$.

Now we assume that $\mathrm{Im}\zeta_0<0$. In this case, we have $c_1=d_2=0$ in (1.17),(1.18), $a^*(\zeta_0^*)=0$ and

$$\overline{\mu}(x,\zeta_0)=-b^*(\zeta_0^*)\nu(x,\zeta_0).$$

Since ζ_0^* is a zero of $a(\zeta)$ in the upper half plane, there must be a bound state $\mu(x,\zeta_0^*)$ and a dependency constant $\tilde{\gamma}_{r,0}$ corresponding to ζ_0^*. By aid of the symmetry property (1.9) of $\Omega(x,\zeta)$

$$\sigma_y\overline{\mu}^{\,*}(x,\zeta_0)=-i\mu(x,\zeta_0^*),\quad \sigma_y\nu^*(x,\zeta_0)=i\overline{\nu}(x,\zeta_0^*).$$

Thus

$$b(\zeta_0^*)=\tilde{\gamma}_{r,0},$$

which is compatible with (1.19). Hence, dependency constant of bound state for $\zeta_0\in\mathbb{C}^-$ can be obtained from bound state for $\zeta_0^*\in\mathbb{C}^+$.

In short, if $\psi(x,\zeta_0)$ is a bound state, then ζ_0 or ζ_0^* must be zero of $a(\zeta)$ in $\mathbb{C}^+$. Conversely, if ζ_0 is a zero of $a(\zeta)$ in $\mathbb{C}^+$, then there are two bound state $\psi(x,\zeta_0)$ and $\psi(x,\zeta_0^*)$ which are related by

$$\psi(x,\zeta_0) = c\sigma_y\psi^*(x,\zeta_0^*),$$

for some constant $c \in \mathbb{C}$. One can obtain the dependency constants $\gamma_{l,r;0}$ from either of them.

In the general case, $\Omega_{\pm\infty} \neq \Lambda$, the eigenvalue ζ_0 may not be zero of $a(\zeta)$. We will discuss this in Section 3.4.

Suppose that

$$a(\zeta) \to 1 \quad \text{as } |\zeta| \to \infty,\ \zeta \in \overline{\mathbb{C}^+}, \tag{1.20}$$

where $\overline{\mathbb{C}^+}$ is the closure of $\mathbb{C}^+$. Since $a(\zeta)$ is an entire function, the number of zeros for $a(\zeta)$ in $\mathbb{C}^+$ should be finite, say N. The normalizing constants $C_{l,r;n}$ are defined by

$$C_{l,n} = \frac{b^*(\zeta_n^*)}{\dot{a}(\zeta_n)}, \quad C_{r,n} = \frac{b(\zeta_n)}{\dot{a}(\zeta_n)},$$

assuming that ζ_n is a simple zero. Here $\dot{}$ denotes the derivative with respect to given variable. The bound state data consists of $\{\zeta_n, C_{l,n}, C_{r,n}\}_{n=1}^N$. The assumption (1.20) will be verified for a specific case in Section 2.2.

We remark that the decay condition (1.6) can be weakened according to the structure of $\Omega(x,\zeta)$ to have analyticity of $a(\zeta)$ in $\mathbb{C}^+$ only. In this case $b(\zeta)$ may not be defined in $\mathbb{C}^+$. Then $C_{l,n}$ and $C_{r,n}$ can be understood as

$$C_{l,n} = \frac{\gamma_{l,n}}{\dot{a}(\zeta_n)}, \quad C_{r,n} = \frac{\gamma_{r,n}}{\dot{a}(\zeta_n)},$$

and the generic condition (1.15) is essential for the coefficient $\Omega(x,\zeta)$ in (1.1) to have only finitely many bound states.

Let

$$A(\zeta) = a(\zeta)\prod_{n=1}^N \frac{\zeta-\zeta_n^*}{\zeta-\zeta_n}.$$

Then $A(\zeta)$ is an entire function and has no zeroes on $\overline{\mathbb{C}^+}$ assuming (1.15), thus $\log A(\zeta)$ and $\log A^*(\zeta^*)$ are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively. Moreover $\log A(\zeta) \to 0$ as $\zeta \to \infty$ in $\mathbb{C}^+$

due to the assumption (1.20). By the Cauchy integral formula, for $\zeta \in \overline{\mathbb{C}^+}$,

$$\begin{aligned}\log A(\zeta + i\varepsilon) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log A(\xi)}{\xi - \zeta - i\varepsilon} d\xi,\\ 0 &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log A^*(\xi)}{\xi - \zeta - i\varepsilon} d\xi.\end{aligned}$$

By adding these,

$$\begin{aligned}\log A(\zeta + i\varepsilon) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log |a(\xi)|^2}{\xi - \zeta - i\varepsilon} d\xi,\\ &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 + |L(\xi)|^2)}{\xi - \zeta - i\varepsilon} d\xi.\end{aligned}$$

Since $a(\zeta)$ is continuous,

$$a(\zeta) = \exp\left(-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log(1 + |L(\xi)|^2)}{\xi - \zeta - i0^+} d\xi\right) \prod_{n=1}^{N} \frac{\zeta - \zeta_n}{\zeta - \zeta_n^*}, \quad \zeta \in \overline{\mathbb{C}^+}. \tag{1.21}$$

This spectral representation allows the scattering data to be defined in serval ways. Here, we mainly consider

$$\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N, \quad \{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N, \tag{1.22}$$

as left and right scattering data respectively. As mentioned earlier, the direct scattering problem is to find the scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$ or $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$ from (1.1) and the inverse scattering problem is to seek a coefficient $\Omega(x, \zeta)$ from $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$ or $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$.

1.2 Inverse Scattering Transformation

In this section, we briefly review the Inverse Scattering Transformation (IST) which gives a motivation for the scattering problem (1.1) for certain choices of $\Omega(x, \zeta)$.

The Schrödinger equation with $u(x, t)$,

$$-\psi_{xx}(x, \zeta; t) + u(x, t)\psi(x, \zeta; t) = \zeta^2 \psi(x, \zeta; t), \tag{1.23}$$

gives a time dependent scattering problem (1.1) by setting

$$\Psi(x, \zeta; t) = \begin{pmatrix} \psi_x(x, \zeta; t) \\ \psi(x, \zeta; t) \end{pmatrix}, \quad \Omega(x, \zeta; t) = \begin{pmatrix} 0 & -\zeta^2 + u(x, t) \\ 1 & 0 \end{pmatrix}.$$

By the similar argument discussed in Section 1.1, one can define the (right) scattering data $\mathbf{S}(t) = \{R(\zeta;t), \zeta_n(t), C_{l,n}(t)\}_{n=1}^{N}$. It is well known that if $u(x,t)$ is in the Faddeev class, i.e. $\displaystyle\int_{-\infty}^{\infty} |u(x,t)|(1+|x|)dx$ is finite for fixed t, then $u(x,t)$ is uniquely determined from its scattering data. For more detail, see e.g. [18], [26] and references therein.

Now, consider the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.24}$$

subject to the initial condition

$$u(x,0) = u_0(x),$$

where $u_0(x)$ decays sufficiently rapidly as $|x| \to \infty$. The IST gives the relation of u, solution to the KdV equation and the scattering data $\mathbf{S}(t)$ for the Schrödinger equation. The KdV equation can be solved in the following manner,

$$\begin{tikzcd}
\boxed{u(x,0)} \arrow[r, "\text{direct scattering problem}"] \arrow[d, swap, "\text{solution of evolution eqn.}"] & \boxed{\mathbf{S}(0)} \arrow[d, "\text{time evolution}"] \\
\boxed{u(x,t)} & \boxed{\mathbf{S}(t)} \arrow[l, swap, "\text{inverse scattering problem}"]
\end{tikzcd}$$

Figure 1.1 Inverse Scattering Transformation

(1) Solve the direct scattering problem (1.23) with $u(x,0)$ to obtain the scattering data $\mathbf{S}(0)$.

(2) Compute the time evolution of scattering data $\mathbf{S}(t)$, which consists of

$$R(\zeta;t) = R(\zeta;0)e^{8i\zeta^3 t}, \quad \zeta_n(t) = \zeta_n(0), \quad C_{l,n}(t) = C_{l,n}(0)e^{-8i\zeta_n^3 t}. \tag{1.25}$$

(3) Solve the inverse scattering problem for $u(x,t)$ from $\mathbf{S}(t)$.

This method was first presented by Gardner, Greene, Kruskal, and Miura in 1967 ([28]). Soon after their work, Lax generalized their idea by introducing 'Lax pair' in [40]. The Lax pair consists of two linear operators $\mathbf{L}$ and $\mathbf{M}$. Consider two linear equations

$$\begin{aligned} \mathbf{L}\phi &= \lambda\phi, && (1.26) \\ \phi_t &= \mathbf{M}\phi. && (1.27) \end{aligned}$$

The first equation is the scattering problem, and the second one governs the time evolution of scattering data. The corresponding non-linear evolution equation is given by

$$\mathbf{L}_t + [\mathbf{L}, \mathbf{M}] = 0, \tag{1.28}$$

where $[\mathbf{L}, \mathbf{M}] = \mathbf{LM} - \mathbf{ML}$. Let

$$\lambda = \zeta^2, \quad \mathbf{L} = -\partial_x^2 + u, \quad \mathbf{M} = -4\partial_x^3 + 6u\partial_x + 3u_x.$$

Then, one can derive (1.25) from (1.27) and the KdV equation (1.24) from (1.28).

Ablowitz, Kaup, Newell and Segur developed a more general scheme, now called the 'AKNS method', in [2], [3]. Consider two linear equations

$$\begin{aligned} \phi_x &= \mathbf{X}\phi, && (1.29) \\ \phi_t &= \mathbf{T}\phi, && (1.30) \end{aligned}$$

where $n \times n$ matrices $\mathbf{X}$ and $\mathbf{T}$ satisfy

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = 0. \tag{1.31}$$

The operator $\mathbf{X}$, in general, contains a spectral parameter ζ and (1.29) represents a scattering problem as does (1.26). Similarly to the operator $\mathbf{M}$ in the Lax pair, $\mathbf{T}$ involves a time evolution of scattering data. The evolution equation corresponding to the given $\mathbf{X}$ and $\mathbf{T}$ can be extracted from the compatibility condition (1.31). In order to obtain the KdV equation, choose

$$\mathbf{X} = \begin{pmatrix} 0 & -\lambda + u \\ 1 & 0 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} u_x & -4\lambda^2 + 2\lambda u + 2u^2 - u_{xx} \\ 4\lambda + 2u & -u_x \end{pmatrix}.$$

For the detailed derivation of the KdV equation via the Lax pair and the AKNS method, refer to [1] or [8]. Several other non-linear evolution equations can be derived via the Lax pair or the AKNS method as well. See [1] and [17] for details.

In this work, we focus on two scattering problems which arise in connection with the cubic Schrödinger equation and the Landau-Lifschitz equations.

- *Cubic Schrödinger equation*

  The cubic Schrödinger equation,

  $$-is_t + s_{xx} + 2|s|^2 s = 0 \tag{1.32}$$

  can be rewritten in the following matrix form,

  $$S_t - i\Lambda S_{xx} - 2i\Lambda S^3 = 0, \tag{1.33}$$

  where

  $$S(x,t) = \begin{pmatrix} 0 & s^*(x,t) \\ s(x,t) & 0 \end{pmatrix}.$$

  Indeed, the complex conjugate of (1.32),

  $$is_t^* + s_{xx}^* + 2|s|^2 s^* = 0,$$

  and $S^2 = |s|^2 I$ implies

  $$i\Lambda S_t + S_{xx} + 2S^2 S = 0,$$

  which is equivalent to (1.33). Define the AKNS operators

  $$\mathbf{X} = i\zeta\Lambda + \Lambda S, \tag{1.34}$$

  $$\mathbf{T} = i\Lambda S^2 - 2i\zeta^2\Lambda + iS_x - 2\zeta\Lambda S. \tag{1.35}$$

  Then $\phi_x = \mathbf{X}\phi$ gives the scattering problem

  $$\phi_x = i\zeta\Lambda\phi + \Lambda S\phi, \tag{1.36}$$

  that is,

  $$\Omega(x,\zeta;t) = i\zeta\Lambda + \Lambda S(x,t),$$

  in the time dependent scattering problem of (1.1). Simple calculations can show that the AKNS equation (1.31) represents the cubic Schrödinger equation (1.33). The time evolution of the scattering data is governed by (1.30) as well. We assume that $s(x,t) \to 0$ as $|x| \to \infty$ sufficiently fast, i.e.

$$\Omega_{\pm\infty}(\zeta;t) = i\zeta\Lambda.$$

We write the solution ϕ of (1.36) which is proportional to the Jost solution μ as

$$\phi(x,t) = f(t)\mu(x;t).$$

As $x \to \infty$, obviously $\phi \to f(t)\begin{pmatrix} e^{i\zeta x} \\ 0 \end{pmatrix}$. From (1.30) with (1.35), we obtain $f(t) = e^{-2i\zeta^2} f(0)$. On the other hand, (1.12) implies

$$\phi(x,t) \to f(t)a(\zeta;t)\begin{pmatrix} e^{i\zeta x} \\ 0 \end{pmatrix} + f(t)b(\zeta;t)\begin{pmatrix} 0 \\ e^{-i\zeta x} \end{pmatrix},$$

as $x \to -\infty$. Substitution in

$$\phi_t = -2i\zeta^2\Lambda\phi$$

which is the limit of (1.30), yields

$$\begin{aligned} f(t)a(\zeta;t) &= e^{-2i\zeta^2 t} f(0)a(\zeta;0), \\ f(t)b(\zeta;t) &= e^{2i\zeta^2 t} f(t)b(\zeta;0), \end{aligned}$$

or,

$$\begin{aligned} a(\zeta;t) &= a(\zeta;0), \\ b(\zeta;t) &= b(\zeta;0)e^{4i\zeta^2 t}. \end{aligned}$$

The time evolution of the scattering data is thus given by

$$R(\zeta;t) = R(\zeta;0)e^{-4i\zeta^2 t}, \quad \zeta_n(t) = \zeta_n(0), \quad C_{l,n}(t) = C_{l,n}(0)e^{-4i\zeta_n^2 t}. \tag{1.37}$$

Note that the time independent scattering problem (1.36) is referred as Zakharov-Shabat scattering problem (ZSSP) after they derived (1.36) via the Lax pair approach in [55].

- *Landau-Lifschitz equation*

The Landau-Lifshitz (LL) equation,

$$\overline{Q}_t = \overline{Q} \times \overline{Q}_{xx} + \overline{Q} \times J\overline{Q} \tag{1.38}$$

describes the dynamics of a ferromagnet. Here,

$$\begin{aligned} J &= \operatorname{diag}(J_1, J_2, J_3), \\ \overline{Q} &= (q_1, q_2, q_3), \quad q_1^2 + q_2^2 + q_3^2 = 1, \end{aligned}$$

and we assume that

$$\overline{Q} \to (0, 0, 1) \quad \text{as } x \to \pm\infty.$$

The matrix J is related to the magnetic anisotropy, for example, isotropy, anisotropy of easy axis and easy plane which are defined by $J_1 = J_2 = J_3$, $J_1 = J_2 < J_3$ and $J_1 = J_2 > J_3$ respectively. It has been attempted to solve the LL equation by the IST in last several decades. Takhtajan ([51]) and Fogedby ([25]) solved the isotropic model of the LL equation and Sklyanin introduced the Lax pair for the anisotropic case in terms of the elliptic function ([50], see also [33] and references therein). In this thesis, we follow the AKNS formulation for the anisotropy of easy axis described in [15].

$$\mathbf{X} = i\zeta \begin{pmatrix} q_3 & q* \\ q & -q_3 \end{pmatrix} - \beta \begin{pmatrix} 0 & q^* \\ -q & 0 \end{pmatrix}, \tag{1.39}$$

$$\mathbf{T} = i \begin{pmatrix} 2(\zeta^2 + \beta^2)q_3 + \zeta\tau_3 & (\zeta + i\beta)(2\zeta q^* + \tau^*) \\ (\zeta - i\beta)(2\zeta q + \tau) & -2(\zeta^2 + \beta^2)q_3 - \zeta\tau_3 \end{pmatrix}, \tag{1.40}$$

where

$$4\beta^2 = J_3 - J_1, \quad q = q_1 + q_2 i, \quad (\tau_1, \tau_2, \tau_3) = \overline{Q} \times \overline{Q}_x, \quad \tau = \tau_1 + \tau_2 i.$$

Then (1.31) implies (1.38). It is not hard to show that the time evolution of the scattering data is governed by

$$R(\zeta; t) = R(\zeta; 0)e^{-4i(\zeta^2+\beta^2)t}, \quad \zeta_n(t) = \zeta_n(0), \quad C_{l,n}(t) = C_{l,n}(0)e^{-4i(\zeta_n^2+\beta^2)t}. \tag{1.41}$$

Table 1.2 Scattering problems and evolution equations

scattering prob.	**evolution eqn.**	**time evolution of data**	**references**
ZSSP	(1.32)	(1.37)	[55]
LLSP	(1.38), $\beta = 0$	(1.41), $\beta = 0$	[51], [25]
ALLSP	(1.38), $\beta \neq 0$	(1.41), $\beta \neq 0$	[15]

We refer to $\psi_x = \mathbf{X}\psi$ with (1.39) for non-zero β as the anisotropic model of the Landau-Lifschitz scattering problem (ALLSP) and the Landau-Lifschitz scattering problem (LLSP) for $\beta = 0$, which represents the isotropic model.

1.3 Overview

A common theme in this work is the inverse scattering problem under an assumption that the coefficients to be determined are of half line support, i.e. they are equal to zero on a half line. There are several reasons why this case is of special interest.

- Such cases can arise on natural physical grounds. This is very common in the case of Schrödinger or acoustic scattering, and more recent work, for example, on coupled mode wave propagation and design of fiber optic devices leads to similar support assumptions for the Zakharov-Shabat problem ([49]).

- In the case of reasonable rapid decay it can be a good approximation to assume the coefficient has half line (or even compact) support.

- Some exact theory for the inverse scattering problem with no support restriction can be developed by means of corresponding half line support problems (see Chapter 4).

- There are special features of the inverse scattering theory which hold in the half line support case which do not hold in general (see Section 2.5).

We remark that a half line support restriction does not make it a half line inverse problem, as this would usually be taken to mean that there is physical boundary point, and boundary condition to be imposed at this point.

This thesis is organized as follows. In Chapter 2, we discuss the ZSSP on the half line. One of the efficient methods to recover the coefficient is the time domain approach. We suggest several computational algorithms to solve the ZSSP based on the time domain problem. Also we show that the reflection coefficient alone uniquely determines the coefficient if the coefficient is supported in the half line. We provide a numerical algorithm as well as an analytic proof.

We define a transformation $\mathfrak{F}$ from the LLSP to the ZSSP in Chapter 3 for a smooth Q. By introducing the step-like coefficient Q, we can define the inverse transformation, i.e. the transformation from the ZSSP to the LLSP, as well. Then we can adapt the more well known theory for the ZSSP to the LLSP. The transformation $\mathfrak{F}$ can be also extended to the ALLSP.

In the previous chapters, we assume that the coefficient is supported in the half line. This support restriction is relaxed in Chapter 4. The whole line problem can be split into two half line problems. We discuss how the scattering data for half line problems can be extracted from the data for the whole line problem. From a numerical point of view, this method reduces the computational cost. The transformation $\mathfrak{F}$ is also well defined in the whole line by the splitting method and step-like coefficients .

Due to the transformation $\mathfrak{F}$, we can show that the coefficient Q in the LLSP is uniquely determined from its scattering data as long as Q is smooth enough. However, if Q has a jump, then the transformation is not valid any more. In Chapter 5, we focus on the LLSP with discontinuous coefficients . An example for the non-uniqueness for the LLSP is given, and some uniqueness theorem with restricted conditions are stated via time domain approach.

In the last chapter, we address open problems and future works.

CHAPTER 2. Zakharov-Shabat scattering problem on the half line

2.1 Introduction

Since Zakharov and Shabat introduced ZSSP in [55], analytic theories as well as numerical methods to reconstruct $s(x)$ from its scattering data have been studied in various directions. Recall the ZSSP

$$\phi_x = i\zeta\Lambda\phi + \Lambda S\phi, \tag{2.1}$$

where,

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & s^* \\ s & 0 \end{pmatrix}.$$

We seek $s(x)$ from standard scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$, or $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$, or equivalently $\{L(\zeta), T(\zeta)\}$, $\{R(\zeta), T(\zeta)\}$ defined in Section 1.1. One of the well known techniques for this inverse scattering problem is using a linear integral equation, the Gel'fand-Levitan-Marchenko (GLM) equation.

For the Jost solutions $J^+ = [\mu\ \overline{\mu}], J^- = [\nu\ \overline{\nu}]$, let

$$J^+(x,\zeta) = e^{i\zeta\Lambda x} + \int_x^\infty \mathcal{K}(x,z)e^{i\zeta\Lambda z}dz, \tag{2.2}$$

$$J^-(x,\zeta) = e^{i\zeta\Lambda x} + \int_{-\infty}^x \mathcal{N}(x,z)e^{i\zeta\Lambda z}dz, \tag{2.3}$$

where $\mathcal{K} = [K\ \overline{K}]$ and $\mathcal{N} = [N\ \overline{N}]$ for two component column vectors $K, \overline{K}, N$, and $\overline{N}$. The crucial point in the representations (2.2) and (2.3) is that the kernels, $\mathcal{K}$ and $\mathcal{N}$ are independent of the spectral parameter ζ. Substitution of (2.2), (2.3) into (2.1) and integration by parts give the following equations for the kernels K and N.

$$K_x(x,y) + \Lambda K_z(x,y) - \Lambda S(x)K(x,y) = 0, \quad x < y, \tag{2.4}$$

$$N_x(x,y) + \Lambda N_z(x,y) - \Lambda S(x)N(x,y) = 0, \quad x > y, \tag{2.5}$$

subject to

$$K^{(2)}(x,x) = -\frac{1}{2}s(x), \quad \lim_{y\to\infty} K(x,y) = 0, \tag{2.6}$$

$$N^{(2)}(x,x) = -\frac{1}{2}s(x), \quad \lim_{y\to-\infty} N(x,y) = 0. \tag{2.7}$$

From the standard technique of characteristics, one can show that there are unique solutions K and N of (2.4)-(2.7). Similarly, one can construct Goursat problems for $\overline{K}$ and $\overline{N}$ to show the existence and uniqueness. Alternately, we can use a symmetry property of the kernels. Recall that the symmetry of $\Omega(x,\zeta)$, (1.9) gives

$$\overline{\mu}(x,\zeta) = -i\sigma_y \mu^*(x,\zeta^*), \quad \overline{\nu}(x,\zeta) = -i\sigma_y \nu^*(x,\zeta^*),$$

which implies

$$\overline{K} = -i\sigma_y K^*, \quad \overline{N} = -i\sigma_y N^*,$$

together with (2.2) and (2.3). Picard's method is another way to show the existence and uniqueness of the kernels $\mathcal{K}, \mathcal{N}$, see [52] for this approach.

In order to derive the GLM equation, we rewrite (1.12) as

$$(T(\zeta)-1)\mu(x,\zeta) = \nu(x,\zeta) + L(\zeta)\overline{\nu}(x,\zeta) - \mu(x,\zeta). \tag{2.8}$$

Operate on this equation with $\frac{1}{2\pi}\int_{\mathbb{R}} \cdot \, e^{-i\zeta y} d\zeta$ for $x>y$, then with (2.2) and (2.3)

$$\frac{1}{2\pi}\int_{\mathbb{R}} (T(\zeta)-1)\mu(x,\zeta) e^{-i\zeta y} d\zeta \tag{2.9}$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{-\infty}^{x} \mathcal{N}(x,z) \begin{pmatrix} e^{i\zeta(z-y)} \\ 0 \end{pmatrix} dz d\zeta \tag{2.10}$$

$$+ \frac{1}{2\pi}\int_{\mathbb{R}} L(\zeta) \begin{pmatrix} 0 \\ e^{-i\zeta(x+y)} \end{pmatrix} d\zeta \tag{2.11}$$

$$+ \frac{1}{2\pi}\int_{\mathbb{R}} L(\zeta) \int_{-\infty}^{x} \mathcal{N}(x,z) \begin{pmatrix} 0 \\ e^{-i\zeta(z+y)} \end{pmatrix} dz d\zeta \tag{2.12}$$

$$- \frac{1}{2\pi}\int_{\mathbb{R}}\int_{x}^{\infty} \mathcal{K}(x,z) \begin{pmatrix} e^{i\zeta(z-y)} \\ 0 \end{pmatrix} dz d\zeta. \tag{2.13}$$

Now, simplify the above integrals term by term. First, since $a(\zeta)$ and $\mu(x,\zeta)$ are analytic in $\mathbb{C}^+$, (2.9) will be

$$-\frac{1}{2\pi}\lim_{r\to\infty}\int_{C_r}(T(\zeta)-1)\mu(x,\zeta)e^{-i\zeta y}d\zeta + i\sum \mathrm{Res}_{\mathbb{C}^+}(T(\zeta)-1)\mu(x,\zeta)e^{-i\zeta y},$$

where, C_r is the upper half circle centered at zero with radius r. Note that for $\zeta \in \mathbb{C}^+$

$$\mu(x,\zeta) = e^{i\zeta x}\begin{pmatrix} 1+o(1) \\ o(1) \end{pmatrix}, \quad \text{as } |\zeta|\to\infty$$

and

$$|T(\zeta)-1| \to 0, \quad \text{as } |\zeta|\to\infty$$

for a sufficiently fast decaying $s(x)$. For details, refer to Proposition 2.2.1. Since we assume $x > y$, the integral term converges to 0. Suppose $a(\zeta)$ has only simple zeros, $\{\zeta_n\}_{n=1}^N$ in the upper half plane. Then

$$\sum \mathrm{Res}_{\mathbb{C}^+}(T(\zeta)-1)\mu(x,\zeta)e^{-i\zeta y} = \sum_n \frac{1}{\dot{a}(\zeta_n)}\mu(x,\zeta_n)e^{-i\zeta_n y},$$

The zero ζ_n of $a(\zeta)$ is an eigenvalue, thus we have

$$\mu(x,\zeta_n) = \gamma_{r,n}\overline{\nu}(x,\zeta_n),$$

as we derived in Section 1.1. Now use the representation (2.3) again. Then (2.9) will be

$$i\sum_n \frac{\gamma_{r,n}}{\dot{a}(\zeta_n)}\begin{pmatrix} 0 \\ e^{-i\zeta_n(x+y)} \end{pmatrix} + i\sum_n \frac{\gamma_{r,n}}{\dot{a}(\zeta_n)}\int_{-\infty}^{x}\overline{N}(x,z)e^{-i\zeta_n(z+y)}dz. \tag{2.14}$$

Next, we define the Fourier transform by

$$\widehat{f}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\zeta)e^{-i\zeta x}d\zeta.$$

Then, the second integral, (2.11) in the right hand side will be $\widehat{L}(x+y)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and (2.12) is converted to

$$\int_{-\infty}^{x}\overline{N}(x,z)\widehat{L}(z+y)dz.$$

The property of the δ-function

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\zeta x} d\zeta$$

makes (2.10) to $N(x, y)$ and (2.13) vanish. Together with (2.14), we obtain the GLM equation

$$N(x,y) + M(x+y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{x} \overline{N}(x,z) M(z+y) dz = 0, \quad x > y, \tag{2.15}$$

where,

$$M(x) = \widehat{L}(x) - i \sum_{n=1}^{N} \frac{\gamma_{r,n}}{\dot{a}(\zeta_n)} e^{-i\zeta_n x} = \widehat{L}(x) - i \sum_{n=1}^{N} C_{r,n} e^{-i\zeta_n x}. \tag{2.16}$$

Similarly, one can obtain the GLM equation for K and $\overline{K}$ from the right scattering data,

$$\overline{K}(x,y) + \tilde{M}(x+y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{x}^{\infty} K(x,z) \tilde{M}(z+y) dz = 0, \quad x < y, \tag{2.17}$$

where,

$$\tilde{M}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} R(\zeta) e^{i\zeta x} d\zeta - i \sum_{n=1}^{N} C_{l,n} e^{i\zeta_n x}. \tag{2.18}$$

The existence and uniqueness of a solution to the GLM equation can be examined by use of the Fredholm alternative, see [5] and [52]. Beals and Coifman ([12], [13]) approached this problem via the Riemann-Hilbert problem, see also [56] for this approach.

Finally, the coefficient $s(x)$ will be extracted from (2.6), (2.7). That is,

$$N^{(2)}(x,x) = -\frac{1}{2} s(x), \quad K^{(2)}(x,x) = -\frac{1}{2} s(x).$$

For numerical algorithms to recover coefficient $s(x)$, refer to [46] and references in this paper. The case of a compactly supported $s(x)$ without bound states was investigated by Rakesh via a contraction mapping in a related time domain problem ([45]).

In this chapter, we restrict the support of $s(x)$ to the right half line and we consider left scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^{N}$, or $\{L(\zeta), T(\zeta)\}$. A time domain inverse problem which is equivalent to the ZSSP, (2.1) is derived in Section 2. In Section 3, we consider $\{L(\zeta), T(\zeta)\}$ as a scattering data for a compactly supported $s(x)$. However, we can show that $L(\zeta)$ alone can reconstruct $s(x)$ if it is supported in the right half line. That is, we can show the uniqueness of the ZSSP without bound state information. Analytic proofs can be found in Section 4, and one of the efficient numerical algorithms, a Darboux kind of transformation is introduced in Section 5. In the last section, we suggest several numerical algorithms and examples for the time domain ZSSP.

2.2 Time domain approach

Assume that $s(x)$ is integrable, that is, $s(x) \in L^1(\mathbb{R})$. Although this assumption is much weaker than (1.6), one can show that the Jost solutions are well defined and $a(\zeta)$ and $b(\zeta)$, the components of transition matrix $\mathcal{T}$, are analytic in the upper and lower half plane respectively and continuous on the real axis ([4]). The proof is based on the Neumann series of the Jost solutions as we discussed in Section 1.1.

We want to develop a hyperbolic system corresponding to (2.1). For this end, let us state the following proposition about the asymptotic behavior of $a(\zeta)$.

Proposition 2.2.1. *Suppose that $s(x) \in L^1(\mathbb{R})$. Then for $\zeta \in \overline{\mathbb{C}^+}$*

$$\mu(x,\zeta) = e^{i\zeta x}\begin{pmatrix} 1+o(1) \\ o(1) \end{pmatrix}, \tag{2.19}$$

$$\overline{\nu}(x,\zeta) = e^{-i\zeta x}\begin{pmatrix} o(1) \\ 1+o(1) \end{pmatrix}, \tag{2.20}$$

and,

$$a(\zeta) = 1 + O(\frac{1}{\zeta}), \tag{2.21}$$

as $|\zeta| \to \infty$.

Proof. We rewrite (1.3) as

$$\mu(x,\zeta)e^{-i\zeta x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\infty \begin{pmatrix} 0 & -s^*(y) \\ s(y)e^{-i\zeta(2x-2y)} & 0 \end{pmatrix} \mu(y,\zeta)e^{-i\zeta y}dy. \tag{2.22}$$

From the similar argument in Section 1.1, it is easy to show $\mu(x,\zeta)$ is uniformly bounded if $s(x)$ decays sufficiently fast, e.g. (1.6). This condition, however, can be weakened to $s(x) \in L^1$. Indeed, one can show that $\mu(x,\zeta)$ is uniformly bounded in $\mathbb{R} \times \overline{\mathbb{C}^+}$, see [4] for details. The Riemann-Lebesgue lemma yields

$$\int_x^\infty s(y)e^{-i\zeta(2x-2y)}\mu^{(1)}(y,\zeta)e^{-i\zeta y}dy = o(1). \tag{2.23}$$

Thus,

$$\mu^{(2)} = o(1).$$

Here, $\mu^{(1)}, \mu^{(2)}$ are components of μ. Substitution of the above representation into the first row of (2.22) gives (2.19). Similarly, we can show (2.20) from (1.4).

To obtain (2.21), we define the Wronskian, $W(u,v)$ for two component vectors u, v as

$$W(u,v) = \det[u\ v].$$

Then obviously, $W(\overline{\nu},\overline{\nu}) = 0$. Since $\mathrm{trace}(i\zeta\Lambda + \Lambda S) = 0$, $W(\mu,\overline{\nu})$ and $W(\nu,\overline{\nu})$ are independent of x by Liouville's formula. Furthermore, $W(\nu,\overline{\nu}) = 1$ from the asymptotic behavior of J^- as $x \to -\infty$. Together with (1.12)

$$\begin{aligned} W(\mu,\overline{\nu}) &= W(a\nu + b\overline{\nu},\overline{\nu}) && (2.24)\\ &= a. && (2.25)\end{aligned}$$

Assuming $s(x) \in W^{1,1}(\mathbb{R})$,

$$\mu(x,\zeta) = e^{i\zeta x}\begin{pmatrix} 1 + O(\frac{1}{\zeta}) \\ O(\frac{1}{\zeta}) \end{pmatrix}, \tag{2.26}$$

$$\overline{\nu}(x,\zeta) = e^{-i\zeta x}\begin{pmatrix} O(\frac{1}{\zeta}) \\ 1 + O(\frac{1}{\zeta}) \end{pmatrix}, \tag{2.27}$$

since in this case we have a better approximation than (2.23),

$$\int_x^\infty s(y)e^{-i\zeta(2x-2y)}\mu^{(1)}(y,\zeta)e^{-i\zeta y}dy = O(\frac{1}{\zeta}).$$

(2.21) is given by the representations of (2.26) and (2.27) together with (2.25).

It is not difficult to verify (2.21) for $s(x) \in L^1(\mathbb{R})$ from an inequality,

$$|a(\zeta;s) - a(\zeta;s')| \leq ||s-s'||_1(1+\min\{||s||_1, ||s'||_1\}) \quad \text{for } \zeta \in \overline{C^+}, \tag{2.28}$$

where $a(\zeta;s), a(\zeta;s')$ are the reciprocals of the transmission coefficients corresponding to the coefficients S, S' whose entries are s, s' respectively. Above inequality, (2.28) was introduced in [35] from integral representations of the Jost solutions. □

One might have asymptotic representations of higher order than (2.21) with stronger condition for $s(x)$. For instance, if $s(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$a(\zeta) = 1 + \frac{1}{2i\zeta}||s||_2^2 + O(\frac{1}{\zeta^2}). \tag{2.29}$$

The proof is similar as the proof of Proposition 2.2.1. First show (2.29) for a smooth s_n (or a step function s_n), then approximate s by s_n.

Consider the solution $\varphi(x,\zeta)$ to (2.1) subject to the boundary condition

$$\varphi(x,\zeta) = \begin{pmatrix} 0 \\ T(\zeta)e^{-i\zeta x} \end{pmatrix}, \quad x < 0$$

for the transmission coefficient $T(\zeta)$. Here, we assume that $s(x) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$[1]. With the generic condition (1.15), $T(\zeta) - 1 \in L^2(\mathbb{R})$ due to Proposition 2.2.1. Multiply $e^{-i\zeta\Lambda x}$ to the left of each side of (2.1). Then

$$(e^{-i\zeta\Lambda x}\varphi)_x = e^{-i\zeta\Lambda x}\Lambda S e^{i\zeta\Lambda x}(e^{-i\zeta\Lambda x}\varphi).$$

For convenient notation, let

$$\tilde{\varphi} := e^{-i\zeta\Lambda x}\varphi - \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{S} := e^{-i\zeta\Lambda x}\Lambda S e^{i\zeta\Lambda x}, \tag{2.30}$$

then

$$\tilde{\varphi}_x = \tilde{S}\tilde{\varphi} + \begin{pmatrix} e^{-2i\zeta x}s^* \\ 0 \end{pmatrix}.$$

[1] We can consider $s(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and vanishes on $x < 0$

For $x > 0$,

$$\tilde{\varphi}(x,\zeta) = \int_0^x \tilde{S}(y,\zeta)\tilde{\varphi}(y,\zeta)dy + \begin{pmatrix} \int_0^x e^{-2i\zeta y} s^*(y)dy \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ T(\zeta) - 1 \end{pmatrix}. \tag{2.31}$$

Thus,

$$\begin{aligned} |\tilde{\varphi}(x,\zeta)| &\leq \int_0^x |\tilde{S}(y,\zeta)||\tilde{\varphi}(y,\zeta)|dy + |\int_0^x e^{-2i\zeta y} s^*(y)dy| + |T(\zeta) - 1| \\ &= \int_0^x |s(y)||\tilde{\varphi}(y,\zeta)|dy + |\int_0^x e^{-2i\zeta y} s^*(y)dy| + |T(\zeta) - 1|. \end{aligned}$$

Recall that $|\cdot|$ denotes the matrix and vector L^2 norms. Since we assume that $s(x) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$, the second integral is bounded by a square integrable function in ζ which is independent of x, since $s(y)\chi_{[0,x]} \in L^2$ for a characteristic function $\chi_{[0,x]}$. From the Plancherel theorem, $|| \int_0^x e^{-2i\zeta y} s^*(y)dy||_2^2 \leq c \int_0^x |s|^2 \leq c||s||_2^2$. Thus there exists a function $M(\zeta) \in L^2(\mathbb{R})$ such that the last two terms are bounded by $M(\zeta)$. For the fixed $\zeta \in \mathbb{R}$, Gronwall's inequality yields

$$|\tilde{\varphi}(x,\zeta)| \leq M(\zeta)e^{||s||_1}, \tag{2.32}$$

for all $x \in \mathbb{R}$. This shows that $\tilde{\varphi}(x,\cdot)$ or $\varphi^{(1)}(x,\cdot), \varphi^{(2)}(x,\cdot) - 1$ are square integrable functions on the real line for any x. Moreover,

$$\tilde{\varphi}(x,\zeta) \to \begin{pmatrix} R(\zeta) \\ 0 \end{pmatrix} \quad \text{as } x \to \infty,$$

by the initial condition of φ. From (2.31), for x, x' (say $x' > x$)

$$|\tilde{\varphi}(x',\zeta) - \tilde{\varphi}(x,\zeta)| \leq \int_x^{x'} |s(y)||\tilde{\varphi}(y,\zeta)|dy + \int_x^{x'} |s(y)|dy.$$

By aid of (2.32),

$$|\tilde{\varphi}(x',\zeta) - \tilde{\varphi}(x,\zeta)| \leq (M(\zeta)e^{||s||_1} + 1) \int_x^{x'} |s(y)|dy.$$

Suppose that x, x' are sufficiently large. Then

$$||\tilde{\varphi}(x',\cdot) - \tilde{\varphi}(x,\cdot)||_2 < \varepsilon$$

for an arbitrary $\varepsilon > 0$. This implies that $R(\zeta) \in L^2(\mathbb{R})$, and thus $L(\zeta) \in L^2(\mathbb{R})$ because $|R(\zeta)| = |L(\zeta)|$. Now we state the following lemma from the above argument.

Lemma 2.2.2. *Let $\phi(x,\zeta)$ be the solution to (2.1) subject to the boundary condition*

$$\phi(x,\zeta)=\begin{pmatrix} e^{i\zeta x} \\ L(\zeta)e^{-i\zeta x}\end{pmatrix}, \quad x<0. \tag{2.33}$$

If $s(x)$ is in $L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+)$ and satisfies the condition (1.15), then for any fixed $x\in\mathbb{R}$,

$$\phi^{(1)}(x,\cdot)e^{-i\zeta x}-1, \phi^{(2)}(x,\cdot)e^{i\zeta x}\in L^2(\mathbb{R}).$$

In particular, $L(\zeta)\in L^2(\mathbb{R})$.

We remark that if $s(x)\in L^1(\mathbb{R}^+)\cap L^\infty(\mathbb{R}^+)$ is assumed, then one can show directly $L(\zeta)\in L^2(\mathbb{R})$ by the argument used in the proof of Proposition 2.2.1 and remark below it. Cohen and Kappeler also showed $L(\zeta)$ is bounded, continuous and square integrable if $s(x)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ from integral representations of the Jost solutions in [20].

Due to Lemma 2.2.2, we can define the Fourier transform of $\phi(x,\cdot)$ in the distributional sense. Let

$$\begin{pmatrix} A(x,t) \\ B(x,t)\end{pmatrix}=\frac{1}{2\pi}\int_{\mathbb{R}}\phi(x,\zeta)e^{-i\zeta t}d\zeta,$$

then (2.1) can be transformed to

$$A_x(x,t)+A_t(x,t)=s^*(x)B(x,t), \quad B_x(x,t)-B_t(x,t)=-s(x)A(x,t), \tag{2.34}$$

and the boundary condition (2.33) is changed to

$$A(x,t)=\delta(t-x), \quad B(x,t)=\widehat{L}(t+x), \quad \text{for } x<0.$$

To define a time domain inverse scattering problem, we need an additional side condition. By the propagation of singularity argument (see e.g. [21] or [16]),

$$\begin{aligned} A(x,t) &= \delta(t-x)+A^u(x,t)H(t-x)+A^d(x,t)H(x-t)+\textit{smooth terms}, \\ B(x,t) &= B^u(x,t)H(t-x)+B^d(x,t)H(x-t)+\textit{smooth terms}. \end{aligned}$$

Here, H is the Heaviside function and $\cdot^u$ and $\cdot^d$ are functions defined on $t>x$ and $t<x$ respectively. Substitution in (2.34) yields

$$A^u_x+A^u_t=s^*B^u, \quad B^u_x-B^u_t=-sA^u,$$
$$A^d_x+A^d_t=s^*B^d, \quad B^d_x-B^d_t=-sA^d,$$

on the regions above and below of $t = x$. Matching the coefficients of the most singular terms along the line $t = x$ we get

$$B^u(x, x^+) - B^d(x, x^-) = \frac{1}{2}s(x). \tag{2.35}$$

Thus, we have two systems of equations,

$$A^u_x + A^u_t = s^* B^u, \quad B^u_x - B^u_t = -sA^u, \tag{2.36}$$

$$A^u(0, t) = 0, \quad B^u(0, t) = \widehat{L}(t), \tag{2.37}$$

on $\mathbf{U}_u := \{(x, t) : x > 0, t > x\}$, and on $\mathbf{U}_d := \{(x, t) : x > 0, t < x\}$,

$$A^d_x + A^d_t = s^* B^d, \quad B^d_x - B^d_t = -sA^d, \tag{2.38}$$

$$A^d(0, t) = 0, \quad B^d(0, t) = \widehat{L}(t). \tag{2.39}$$

Above two systems of equations are combined by a 'Jump condition' (2.35).

The derivation of a time domain problem assuming no bound states is well known, see [46], [45] and references in these papers. This time domain problem can be understood as the special case when $A^d = B^d = 0$, which can be justified by the following theorem.

Theorem 2.2.3. *Suppose that $s(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and is supported on the right half line in the ZSSP. If $|a(\zeta)| > 0$ for $\zeta \in \mathbb{R}$, then the following are equivalent.*

(1) There are no bound states.

(2) $T(\zeta) - 1 \in H^2_+(\mathbb{R})$.

(3) $L(\zeta) \in H^2_+(\mathbb{R})$.

(4) For the solution ϕ to (2.1) with boundary condition (2.33),

$$\phi^{(1)}(x, \cdot)e^{-i\zeta x} - 1 \in H^2_+, \quad \phi^{(2)}(x, \cdot)e^{i\zeta x} \in H^2_+ \quad \textit{for any } x \in \mathbb{R}.$$

Here, H^2_+ denotes the Hardy space, i.e.

$$H^2_+ = \{F : F \text{ is analytic on } \mathbb{C}^+, \quad \sup_{y>0} \int |F(x + iy)|^2 dx < \infty\}$$

or equivalently

$$H^2_+ = \{F : F \in L^2(\mathbb{R}), \widehat{F}(t) = 0 \text{ for } t < 0\}.$$

Proof. Suppose that there are no bound states. Then $T(\zeta)-1$ is analytic in $\mathbb{C}^+$ and going to zero like $O(1/\zeta)$ as $|\zeta|\to\infty$ from the Proposition 2.2.1. Thus $T(\zeta)-1\in H_+^2$. Now we claim that $\tilde{\varphi}(x,\cdot)$ defined in (2.30) is also in the Hardy space for any fixed x. Indeed, we can choose $M(\zeta)$ in (2.32) as $|\int_0^x e^{-2i\zeta y}s^*(y)dy|+|T(\zeta)-1|$. Since $\int_0^x e^{-2i\zeta y}s^*(y)dy$ and $T(\zeta)-1$ are in H_+^2,

$$\sup_{\zeta_2>0}\int|\tilde{\varphi}(x,\zeta_1+\zeta_2 i)|^2 d\zeta_1<\infty,$$

from (2.32). Due to the fact that $\varphi^{(1)}(x,\cdot)\to R(\cdot)$ as $x\to\infty$ in L^2, $R(\zeta)\in H_+^2$, and so $L(\zeta)\in H_+^2$. Note that $R(\zeta)$ and $L(\zeta)$ are extended off the real axis by

$$R(\zeta)=\frac{b^*(\zeta^*)}{a(\zeta)},\quad L(\zeta)=\frac{b(\zeta)}{a(\zeta)},$$

if they are defined.

To have (4) from (3), applied the same argument about $\tilde{\varphi}$ to ϕ, the solution to (2.1) with (2.33). The converse is obvious, that is (4) implies (3). If $L(\zeta)$ is analytic in upper half plane, $a(\zeta)$ never vanishes in $\mathbb{C}^+$. Hence there are no bound states. □

The time domain inverse scattering problem in case of no bound states is to seek $s(x)$ from (2.36), (2.37) and

$$B^u(x,x^+)=\frac{1}{2}s(x). \tag{2.40}$$

The uniqueness and existence of this problem was investigated by Rakesh in [45] for a compactly supported $s(x)$ by defining a mapping $\mathfrak{C}$ as

$$\begin{aligned}\mathfrak{C}:\quad L^2[0,X]&\to L^2[0,X]\\ s(x)\quad&\mapsto 2B^u(x,x)\end{aligned}\tag{2.41}$$

where, B^u solves (2.36) and (2.37) in $\mathbf{U}_{u,X}=\{(x,t)\in\mathbf{U}_u : t+x\le 2X\}$ for a given $\widehat{L}(t)\chi_{[0,2X]}$.

Theorem 2.2.4 (Rakesh [45]). *The map $\mathfrak{C}$ is well defined for any $\widehat{L}(t)\chi_{[0,2X]}$, and it has a unique fixed point.*

Indeed, he showed $\mathfrak{C}^p$ is a contraction map for a sufficiently large p. One can easily adapt his result in showing the uniqueness of $s(x)\in\mathcal{S}_0$, where

$$\mathcal{S}_0=\{s(x)\in L^1(\mathbb{R}^+)\cap L^2(\mathbb{R}^+) : |a(\zeta)|>0 \text{ for } \zeta\in\overline{\mathbb{C}^+}\}. \tag{2.42}$$

Recall that $s(x) \in \mathcal{S}_0$ can be understood as $s(x) = 0$ for $x < 0$. Suppose that $s_1(x), s_2(x) \in \mathcal{S}_0$ are solutions of the time domain inverse problem with a fixed $\widehat{L}(t)$. We claim that there exists $M > 0$ such that for any $X \geq M$

$$s_1 \chi_{[0,X]}, s_2 \chi_{[0,X]} \in \mathcal{S}_0. \tag{2.43}$$

Assuming this, we have $s_1(x) = s_2(x)$ for $x < X$ from Theorem 2.2.4. Since X is an arbitrary number which is larger than M, $s_1(x) = s_2(x)$ on $\mathbb{R}$. It is not difficult to verify (2.43) from the inequality (2.28). Since $s_1(x) \in \mathcal{S}_0$ and $a(\zeta) \to 1$ as $|\zeta| \to \infty$ for $\zeta \in \overline{\mathbb{C}^+}$, there exists $\delta_1 > 0$ such that

$$|a(\zeta; s_1)| \geq \delta_1, \quad \zeta \in \overline{C^+}. \tag{2.44}$$

(2.28) guarantees the existence of $M_1 > 0$ such that for any $X_1 \geq M_1$

$$|a(\zeta; s_1) - a(\zeta; s_1 \chi_{[0,X_1]})| \leq \delta_1/2,$$

since $s_1(x) \in L^1$. Together with (2.44), we have

$$|a(\zeta; s_1 \chi_{[0,X_1]})| \geq \delta_1/2 > 0, \quad \zeta \in \overline{C^+}.$$

Similarly, there is M_2 such that for any $X_2 \geq M_2$

$$|a(\zeta; s_2 \chi_{[0,X_2]})| > 0, \quad \zeta \in \overline{C^+}.$$

Thus, (2.43) is valid for $X \geq \max\{M_1, M_2\}$.

To finish the proof of the uniqueness of the ZSSP for $s(x) \in \mathcal{S}_0$, we need to show that the ZSSP is equivalent to the time domain problem, or the existence of the Fourier inverse transform of solutions to the time domain inverse problem. This can be easily shown by Stokes's theorem. One can obtain

$$(|A^u|^2 + |B^u|^2)_x + (|A^u|^2 - |B^u|^2)_t = 0,$$

from (2.36). Integrating this over the region $OABC$ (see Figure 2.1) and Stokes' theorem give

$$\int_{OC} |B^u|^2 = \int_{OA} 2|B^u|^2 + \int_{CB} 2|A^u|^2 + \int_{AB} |A^u|^2 + |B^u|^2,$$

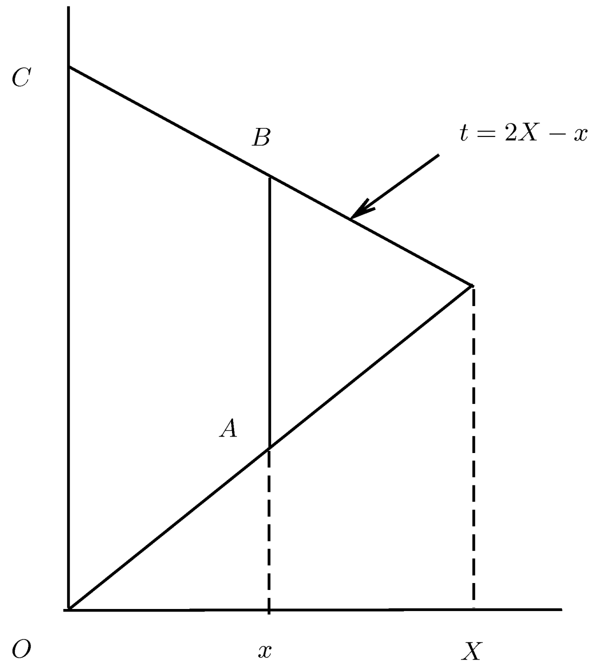

Figure 2.1 Region $OABC$ in t-x plane

or

$$\int_x^{2X-x} |A^u(x,\cdot)|^2 + |B^u(x,\cdot)|^2 \leq \int_0^{2X} |\widehat{L}|^2 \leq ||\widehat{L}||_2^2.$$

Since we assumed no bound states, $A^u(x,t)$ and $B^u(x,t)$ can be extended by 0 for $t < x$. Thus

$$||A^u(x,\cdot)||_2^2 + ||B^u(x,\cdot)||_2^2 \leq ||\widehat{L}||_2^2. \tag{2.45}$$

We summarize the above argument as a theorem.

Theorem 2.2.5. *Suppose that $s(x) \in \mathcal{S}_0$. Then the ZSSP and the time domain inverse problem,* (2.36),(2.37) *and* (2.40) *are equivalent, and the ZSSP has at most one solution. That is, the map*

$$\begin{aligned} \mathfrak{S}: \quad \mathcal{S}_0 &\rightarrow H_+^2 \\ s(x) &\mapsto L(\zeta) \end{aligned}$$

is one to one.

We remark that the range of $\mathfrak{S}$ is a strict subset of H_+^2 since Villarroel et al. showed $\widehat{L} \in L^1$ for $s(x) \in L^1$ in [52]. Even for this class, however, the surjection is unsolved yet.

2.3 ZSSP in the case of compact support

In this section, we discuss the ZSSP with compactly supported $s(x)$ on $[0, X]$ with and without bound states. As mentioned earlier, several numerical algorithms to recover $s(x)$ from the scattering data have been developed. One of the most efficient methods in case of no bound states is a layer stripping method, which will be discussed in Section 2.6. Here, we introduce another method based on the jump condition (2.35) derived in Section 2.2 with $\{L(\zeta), T(\zeta)\}$ as a scattering data.

Since the solution to (2.1) with the boundary condition (2.33) satisfies $\phi(x,\zeta) = \begin{pmatrix} T(\zeta)e^{i\zeta x} \\ 0 \end{pmatrix}$ for $x > X$, the side condition (2.39) can be rewritten as

$$A^d(X,t) = \widehat{T-1}(t-X), \quad B^d(X,t) = 0, \tag{2.46}$$

for $t < X$. A simple change of variables $t' = X - t, x' = X - x$ makes (2.38),(2.46) transform to

$$\begin{aligned} {A'}^d_{x'} + {A'}^d_{t'} &= -s^*(X-x'){B'}^d, & {B'}^d_{x'} - {B'}^d_{t'} &= s(X-x'){A'}^d, \\ {A'}^d(0,t') &= \widehat{T-1}(-t'), & {B'}^d(0,t') &= 0, \end{aligned}$$

on $\mathbf{U}_u$, where ${A'}^d(x',t') := A^d(x,t)$ and ${B'}^d(x',t') := B^d(x,t)$. Remove suffix $'$, then with (2.36),(2.37),

$$A^u_x + A^u_t = s^* B^u, \qquad B^u_x - B^u_t = -sA^u, \tag{2.47}$$

$$A^d_x + A^d_t = -s^*(X-x)B^d, \quad B^d_x - B^d_t = s(X-x)A^d, \tag{2.48}$$

on $\mathbf{U}_u$, and

$$A^u(0,t) = 0, \qquad B^u(0,t) = \widehat{L}(t), \tag{2.49}$$

$$A^d(0,t) = \widehat{T-1}(-t), \quad B^d(0,t) = 0, \tag{2.50}$$

with the jump condition

$$2(B^u(x,x) - B^d(X-x, X-x)) = s(x). \tag{2.51}$$

Proposition 2.3.1. *For fixed scattering data* $\widehat{L}(t), \widehat{T-1}(-t) \in L^2(0,2X)$*, the map*

$$\begin{aligned} \tilde{\mathfrak{C}}: \quad L^2(0,X) &\rightarrow L^2(0,X) \\ s(x) &\mapsto 2(B^u(x,x) - B^d(X-x, X-x)) \end{aligned}$$

is well defined and Lipschitz continuous.

Proof. First, define a reflection operator $\mathcal{R}$ by $\mathcal{R}(s)(x) = s(X-x)$. Obviously $\mathcal{R}$ is a linear isometry. Let $\mathfrak{C}(s)(x) = 2B(x,x)$ from (2.47), (2.49) as in (2.41). Similarly, define $\mathfrak{C}'(s)(x) = 2B^d(x,x)$ from (2.48),(2.50). Then

$$\tilde{\mathfrak{C}}(s) = \mathfrak{C}(s) - \mathcal{R}\mathfrak{C}'\mathcal{R}(-s),$$

which is well defined since B^u, B^d have L^2 traces on any lines segments. Moreover, from Rakesh's result (see Theorem 2.2.4 and [45]) $\mathfrak{C}$ and $\mathfrak{C}'$ are Lipschitz continuous. Let M, M' be the Lipschitz constants. Then for $s_1, s_2 \in L^2(0,X)$

$$\begin{aligned} ||\tilde{\mathfrak{C}}(s_1) - \tilde{\mathfrak{C}}(s_2)||_2 &= ||\mathfrak{C}(s_1) - \mathcal{R}\mathfrak{C}'\mathcal{R}(-s_1) - \mathfrak{C}(s_2) + \mathcal{R}\mathfrak{C}'\mathcal{R}(-s_2)||_2 \\ &\leq ||\mathfrak{C}(s_1) - \mathfrak{C}(s_2)|| + ||\mathcal{R}\mathfrak{C}'\mathcal{R}(-s_1) - \mathcal{R}\mathfrak{C}'\mathcal{R}(-s_2)||_2 \\ &\leq M||s_1 - s_2|| + ||\mathfrak{C}'\mathcal{R}(-s_1) - \mathfrak{C}'\mathcal{R}(-s_2)||_2 \\ &\leq M||s_1 - s_2|| + M'||\mathcal{R}(-s_1) - \mathcal{R}(-s_2)||_2 \\ &= M||s_1 - s_2||_2 + M'||s_1 - s_2||_2 = N||s_1 - s_2||_2. \end{aligned}$$

□

The Lipschitz constant, N depends on $X, \widehat{L}$, and $\widehat{T-1}$. If $N < 1$, then obviously $\tilde{\mathfrak{C}}$ is a contraction mapping. But this is not true in general. Besides, we do not have a local contraction property either, that is, $\tilde{\mathfrak{C}}^p$ need not be a contraction mapping in contrast with $\mathfrak{C}$. Thus, we need a new idea to attack the time domain inverse scattering problem involving bound states.

We define a map Γ_r for a fixed $r \in L^2(0, 2X)$ as

$$\begin{aligned} \Gamma_r : \quad L^2(0,X) \quad &\to L^2(0,2X) \\ s \qquad &\mapsto \quad \overline{r} \end{aligned}$$

where, the relations of $r, \overline{r}$ and s are governed by (2.47), (2.48) with initial conditions

$$A^u(0,t) = 0, \qquad B^u(0,t) = \overline{r}(t), \tag{2.52}$$

$$A^d(0,t) = r(t), \qquad B^d(0,t) = 0, \tag{2.53}$$

and the jump condition (2.51). Then Γ_r is well defined. To show this, first consider (2.48) with (2.53) on $\mathbf{U}_{u,X}$. Recall that $\mathbf{U}_{u,X} = \{(x,t) \in \mathbf{U}_u : t + x \leq 2X\}$. Since $B^d(x,x) \in L^2(0,X)$, from the jump condition

$$B^u(x,x) = \frac{1}{2}s(x) + B^d(X-x, X-x) \in L^2(0,X).$$

Now, we have the following characteristic boundary value problem on $\mathbf{U}_{u,X}$,

$$A^u_x + A^u_t = s^* B^u, \quad B^u_x - B^u_t = -sA^u, \tag{2.54}$$

$$A^u(0,t) = 0, \quad 0 < t < 2X, \tag{2.55}$$

$$B^u(x,x) = \tfrac{1}{2}s(x) + B^d(X-x, X-x), \quad 0 < x < X. \tag{2.56}$$

The uniqueness and existence of (2.54)-(2.56) can be shown by using standard techniques for the system of hyperbolic equations, see e.g. [27, 21].

We seek $s(x)$ from the knowledge of $\{L(\zeta), T(\zeta)\}$ or equivalently $\{\widehat{L}(t), \widehat{T-1}(t)\}$. This problem can be understood as solving the following nonlinear equation.

$$\Gamma_{\widehat{T-1}(-t)}(s) = \widehat{L}. \tag{2.57}$$

One of the well known methods to solve a nonlinear equation is Newton's method. For this, the existence of a linearized map is essential. We define the Frechlet derivative of Γ_r at s in the direction p by

$$D\Gamma_r(s)p = \lim_{h \to 0} \frac{\Gamma_r(s + hp) - \Gamma_r(s)}{h}.$$

Theorem 2.3.2. *For $r \in L^2(0,2X)$, Γ_r is Frechlet differentiable at 0, and*

$$D\Gamma_r(0) : L^2(0,X) \to L^2(0,2X)$$

is given by

$$D\Gamma_r(0)p(x) = \frac{1}{2}p(\frac{x}{2}) + \frac{1}{2}\int_x^{2X} p(\frac{z}{2})r(z-2x)dz. \tag{2.58}$$

For the proof, we define Φ as

$$\begin{aligned} \Phi : \quad L^2(0,X) \times L^2(0,2X) \times L^2(0,2X) \quad &\to \quad L^2(0,X) \\ (s(x), f(t), g(t)) \quad &\mapsto \quad B(x,x) \end{aligned} \tag{2.59}$$

where, $A(x,t)$ and $B(x,t)$ solve (2.47),

$$A_x + A_t = s^* B, \quad B_x - B_t = -sA, \quad \text{in } \mathbf{U}_{u,X} \tag{2.60}$$

subject to

$$A(0,t) = f(t), \quad B(0,t) = g(t). \tag{2.61}$$

Similarly, we define $\overline{\Phi}$ by

$$\begin{aligned} \overline{\Phi} : \quad L^2(0,X) \times L^2(0,2X) \times L^2(0,2X) \quad &\to \quad L^2(0,X) \\ (s(x), f(t), g(t)) \quad &\mapsto \quad A(x, 2X - x) \end{aligned}$$

One can easily show that $\Phi, \overline{\Phi}$ are well defined. Also they have following properties.

Lemma 2.3.3. *For $s \in L^2(0,X), f, \overline{f}, g, \overline{g} \in L^2(0,2X)$, Φ and $\overline{\Phi}$ satisfy the following properties.*

(1) $\Phi(s,f,g) \pm \Phi(s,\overline{f},\overline{g}) = \Phi(s, f \pm \overline{f}, f \pm \overline{g})$.

(2) If $|h| = |k|$ for nonzero $h, k \in \mathbb{C}$,

$$h\Phi(s,f,g) = \Phi(\frac{h}{k}s, kf, hg). \tag{2.62}$$

(3) $||\Phi(s,f,g)||^2 + ||\overline{\Phi}(s,f,g)||^2 = ||g||^2 + ||f||^2$.

(4) $||\Phi(s,0,g) - \Phi(0,0,g)||^2 = ||\Phi(s,0,g) - g(2x)||^2 \le C_1||s||$.

(5) $||\overline{\Phi}(s,0,g) - \overline{\Phi}(0,0,g)||^2 = ||\overline{\Phi}(s,0,g)||^2 \le C_1||s||$.

(6) $||\frac{1}{h}\Phi(hs,f,0) + \int_0^x s(z)f(2x-2z)dz||^2 \le C_2|h|^2$ *for sufficiently small* h.

Here $||\cdot||$ *denotes* L^2 *norm in* $(0,X)$ *or* $(0,2X)$.

Proof. From the definition of the maps $\Phi, \overline{\Phi}$, $(1),(2)$ are obvious. We can obtain

$$(|A|^2 + |B|^2)_x + (|A|^2 - |B|^2)_t = 0,$$

from (2.60). Then Stokes' theorem over $\mathbf{U}_{u,X}$ together with the boundary conditions (2.61) gives (3).

For proofs of $(4),(5)$, consider

$$A'_x + A'_t = 0, \quad B'_x - B'_t = 0, \quad \text{in } \mathbf{U}_{u,X}$$

with

$$A'(0,t) = 0, \quad B'(0,t) = g(t), \tag{2.63}$$

and (2.60) with the same boundary conditions (2.63). Trivially, $\Phi(0,0,g) = g(2x)$ and $\overline{\Phi}(0,0,g) = 0$.

Let $\tilde{A} = A - A', \tilde{B} = B - B'$, then

$$\tilde{A}_x + \tilde{A}_t = s^*B, \quad \tilde{B}_x - \tilde{B}_t = -sA, \tag{2.64}$$

$$\tilde{A}(0,t) = 0, \quad \tilde{B}(0,t) = 0. \tag{2.65}$$

We obtain

$$(|\tilde{A}|^2 + |\tilde{B}|^2)_x + (|\tilde{A}|^2 - |\tilde{B}|^2)_t = 2\mathrm{Re}(s^*B\tilde{A}^* - sA\tilde{B}^*)$$

from (2.64). Again we apply Stokes' theorem over $OABC$ (see Figure 2.1).

$$\int_{OA} 2|\tilde{B}|^2 + \int_{CB} 2|\tilde{A}|^2 + \int_{AB} |\tilde{A}|^2 + |\tilde{B}|^2 = \int\int_{OABC} 2\mathrm{Re}(s^*B\tilde{A}^* - sA\tilde{B}^*). \tag{2.66}$$

Define

$$E(x) = \int_{OA} 2|B|^2 + \int_{CB} 2|A|^2 + \int_{AB} |A|^2 + |B|^2.$$

Then it is easy to see

$$E(x) = ||g||^2,$$

for any $0 \leq x \leq X$. Similarly, define

$$\tilde{E}(x) = \int_{OA} 2|\tilde{B}|^2 + \int_{CB} 2|\tilde{A}|^2 + \int_{AB} |\tilde{A}|^2 + |\tilde{B}|^2.$$

The right hand side of (2.66) can be

$$\begin{aligned} \int\int_{OABC} 2\mathrm{Re}(s^* B\tilde{A}^* - sA\tilde{B}^*) &\leq \int\int_{OABC} |s|(|A|^2 + |B|^2) + |s|(|\tilde{A}|^2 + |\tilde{B}|^2) \\ &\leq \int_0^x |s|E + |s|\tilde{E}. \end{aligned}$$

Thus, we have

$$\tilde{E}(x) \leq ||s||_1||g||^2 + \int_0^x |s|\tilde{E}.$$

So Gronwall's inequality implies

$$\tilde{E}(x) \leq \sqrt{X}||s||||g||^2 e^{\sqrt{X}||s||}.$$

(4) and (5) follow from

$$\begin{aligned} \tilde{E}(X) &= 2\int_0^X |B(x,x) - B'(x,x)|^2 \\ &+ 2\int_0^X |A(x, 2X - x) - A'(x, 2X - x)|^2 \\ &= 2||\Phi(s, 0, g) - \Phi(0, 0, g)||^2 + 2||\overline{\Phi}(s, 0, g) - \overline{\Phi}(0, 0, g)||^2. \end{aligned}$$

Now we prove the last inequality. The identity (2.62) implies $\frac{1}{h}\Phi(hs, f, 0) = \Phi(hs, f/h, 0)$ by setting $k = h$. $\Phi(hs, f/h, 0)$ is $B(x, x)$ satisfying

$$A_x + A_t = (hs)^* B, \quad B_x - B_t = -hsA,$$

$$A(0,t) = f(t)/h, \quad B(0,t) = 0,$$

or

$$(hA)_x + (hA)_t = |h|^2 s^* B, \quad B_x - B_t = -s(hA),$$

$$(hA)(0,t) = f(t), \quad B(0,t) = 0.$$

Let

$$E(x) = \int_{OA} 2|B|^2 + \int_{CB} 2|hA|^2 + \int_{AB} |hA|^2 + |B|^2.$$

Stokes' theorem gives

$$\begin{aligned} E(x) &\leq ||f||^2 + \int\int_{OABC} 2(|h|^2+1)|s||hA||B| \\ &\leq ||f||^2 + \int_0^x (|h|^2+1)|s|E. \end{aligned}$$

Thus, for all $0 \leq x \leq X$

$$E(x) \leq ||f||^2 e^{(|h|^2+1)\sqrt{X}||s||}. \tag{2.67}$$

Consider

$$A'_x + A'_t = 0, \quad B'_x - B'_t = -sA',$$

$$A'(0,t) = f(t), \quad B'(0,t) = 0.$$

It is not difficult to show

$$A'(x,t) = f(t-x), \quad B'(x,t) = -\int_0^x s(z) f(x+t-2z)dz.$$

Thus,

$$\frac{1}{h}\Phi(hs,f,0) + \int_0^x s(z) f(2x-2z)dz = B(x,x) - B'(x,x).$$

Similarly to the proof of (4), (5), we define

$$\begin{aligned} \tilde{A} &= hA - A', \quad \tilde{B} = B - B' \\ \tilde{E}(x) &= \int_{OA} 2|\tilde{B}|^2 + \int_{CB} 2|\tilde{A}|^2 + \int_{AB} |\tilde{A}|^2 + |\tilde{B}|^2. \end{aligned}$$

Here, $\tilde{A}$ and $\tilde{B}$ solve

$$\tilde{A}_x + \tilde{A}_t = |h|^2 s^* B, \quad \tilde{B}_x - \tilde{B}_t = -s\tilde{A},$$

$$\tilde{A}(0,t) = 0, \quad \tilde{B}(0,t) = 0.$$

Again we apply Stokes' theorem.

$$\begin{aligned} \tilde{E}(x) &\leq 2\int\int_{OABC} |h|^2|s||B||\tilde{A}| + |s||\tilde{A}||\tilde{B}| \\ &\leq \int_0^x |h|^2|s|E + \int_0^x (|h|^2+1)|s|\tilde{E}. \end{aligned}$$

Gronwall's inequality together with (2.67) yields

$$\tilde{E}(x) \leq C(X, ||s||, ||f||, |h|)|h|^2.$$

Although C depends on $|h|$, if we assume $|h|$ is small enough, say $|h| < 1$, then C is bounded by some C_2 which is independent of $|h|$. Thus

$$||\tilde{B}(x,x)||^2 \leq \tilde{E}(X) \leq C_2|h|^2.$$

□

Proof of Theorem 2.3.2. For convenience of notation, we omit r from Γ_r if it is obvious. From the definition of the Frechlet derivative

$$D\Gamma(0)p = \lim_{h\to 0} \frac{\Gamma(hp) - \Gamma(0)}{h}.$$

It is not difficult to check that $\Gamma(0) = 0$ from directly solving the equations, and $\Gamma(hp)$ satisfies

$$\frac{1}{2}hp = \Phi(hp, 0, \Gamma(hp)) - \mathcal{R}\Phi(-\mathcal{R}hp, r, 0), \tag{2.68}$$

from the definition of Φ and (2.56). By Lemma 2.3.3 (1) and (2), we can rewrite (2.68) as

$$\Phi(hp, 0, \frac{\Gamma(hp)}{h}) = \frac{1}{2}p + \mathcal{R}\frac{1}{h}\Phi(-h\mathcal{R}p, r, 0). \tag{2.69}$$

Lemma 2.3.3 (6) implies that the second term in the right hand side converges to

$$\begin{aligned} \mathcal{R}\int_0^x \mathcal{R}p(z)r(2x-2z)dz &= \int_0^{X-x} p(X-z)r(2X-2x-2z)dz \\ &= \frac{1}{2}\int_{2x}^{2X} p(\frac{z}{2})r(z-2x)dz, \end{aligned}$$

and

$$||\Phi(hp, 0, \frac{\Gamma(hp)}{h}) - \rho(x)|| \leq C|h|,$$

where

$$\rho(x) = \frac{1}{2}p(x) + \frac{1}{2}\int_{2x}^{2X} p(\frac{z}{2})r(z-2x)dz.$$

Then, for $|h| < 1$

$$
\begin{aligned}
& ||\frac{\Gamma(hp)}{h} - \rho(\frac{x}{2})||^2 \\
= \;& ||\Phi(hp, 0, \frac{\Gamma(hp)}{h} - \rho(\frac{x}{2}))||^2 + ||\overline{\Phi}(hp, 0, \frac{\Gamma(hp)}{h} - \rho(\frac{x}{2}))||^2 \quad \text{by Lemma 2.3.3 (3)} \\
\leq \;& ||\Phi(hp, 0, \frac{\Gamma(hp)}{h}) - \Phi(hp, 0, \rho(\frac{x}{2}))||^2 + C|h| \quad \text{by Lemma 2.3.3 (1), (5)} \\
= \;& ||\Phi(hp, 0, \frac{\Gamma(hp)}{h}) - \rho(x) + \rho(x) - \Phi(hp, 0, \rho(\frac{x}{2}))||^2 + C|h| \\
\leq \;& 2||\Phi(hp, 0, \frac{\Gamma(hp)}{h}) - \rho(x)||^2 + 2||\Phi(hp, 0, \rho(\frac{x}{2})) - \rho(x)||^2 + C|h| \\
< \;& C|h| \quad \text{by (2.70), Lemma 2.3.3 (4).}
\end{aligned}
$$

Thus we prove Theorem 2.3.2. $\square$

We may show that Γ_r is Frechlet differentiable at any s for $s \in L^2(0, X)$. Then Newton's method can be used to solve (2.57) by the following iteration scheme,

$$s_{n+1} = s_n - D\Gamma(s_n)^{-1}(\Gamma(s_n) - \widehat{L}),$$

provided $D\Gamma(s_n)$ is nonsingular and $||s - s_0||$ is small enough. However, $D\Gamma(s_n)$ is quite complicated to compute. Indeed, we have to solve three hyperbolic system of equations per each step. Instead, we consider the following modified Newton's method,

$$
\begin{aligned}
s_{n+1} &= s_n - D\Gamma(0)^{-1}(\Gamma(s_n) - \widehat{L}) && (2.70) \\
&= (I - D\Gamma(0)^{-1}\Gamma)s_n + D\Gamma(0)^{-1}\widehat{L}, && (2.71)
\end{aligned}
$$

where I is an identity map. To show the convergence of (2.71), we have to investigate the injectivity and surjectivity of $D\Gamma(0)$. Although this might be a well known theory, we show here by introducing a weighted norm.

Theorem 2.3.4. *The linearized map of Γ_r at 0 is a one to one map from $L^2(0, X)$ onto $L^2(0, 2X)$ for any $r \in L^2(0, 2X)$.*

Proof. First, we define a weighted L^2 norm for fixed λ as

$$||p||_\lambda^2 = \int_0^X |p(x)|^2 e^{2\lambda x} dx.$$

Obviously, $L^2_\lambda(0,X)$ is equivalent to $L^2(0,X)$.

By scaling, we rewrite the linearized map of Γ_r as

$$\begin{array}{rccl} D\tilde{\Gamma}: & L^2(0,X) & \to & L^2(0,X) \\ & p(x) & \mapsto & p(x) + \displaystyle\int_x^X p(z)r(z-x)dz. \end{array}$$

Consider the following Volterra integral equation of the second kind,

$$p(x) + \int_x^X p(z)r(z-x)dz = f(x). \tag{2.72}$$

To show a bijectivity of $D\tilde{\Gamma}$, it is enough to show that the norm of map $\mathcal{V}$ defined by

$$\begin{array}{rccl} \mathcal{V}: & L^2_\lambda(0,X) & \to & L^2_\lambda(0,X) \\ & p(x) & \mapsto & \displaystyle\int_x^X p(z)r(z-x)dz \end{array}$$

is bounded by a positive constant $C < 1$. Indeed, the uniqueness and existence of a solution (2.72) can be established by the Neumann series ([38]), and this implies a bijectivity of $D\Gamma$.

For any $p(x) \in L^2(0,X)$,

$$\begin{aligned} ||\mathcal{V}p||^2_\lambda &\leq X\int_0^X \int_x^X |p(z)|^2|r(z-x)|^2 e^{2\lambda x} dz dx \\ &= X\int_0^X |p(z)|^2 e^{2\lambda z}\int_0^z |r(z-x)|^2 e^{-2\lambda(z-x)} dx dz \\ &\leq X||r(\cdot)e^{-\lambda(\cdot)}||^2||p||^2_\lambda. \end{aligned}$$

Thus,

$$||\mathcal{V}||_\lambda \leq \sqrt{X}||r(\cdot)e^{-\lambda(\cdot)}||.$$

If we choose λ sufficiently large, then $\mathcal{V}$ is a contraction mapping, thus we prove the theorem.

□

In general, however, $||I - D\Gamma(0)^{-1}\Gamma|| \geq 1$, thus we have to modify (2.71) by using a damping factor ω.

$$\begin{aligned} \overline{s} &:= (I - D\Gamma(0)^{-1}\Gamma)s_n + D\Gamma(0)^{-1}\widehat{L}, \\ s_{n+1} &= s_n + \omega(\overline{s} - s_n). \end{aligned}$$

Eliminating $\overline{s}$ yields

$$s_{n+1} = (I - \omega D\Gamma(0)^{-1}\Gamma)s_n + \omega D\Gamma(0)^{-1}\widehat{L}. \tag{2.73}$$

The convergence of above iteration scheme may follow from the continuously differentiability of Γ. More precisely, we can state this as follows.

Suppose that Γ is bounded in a open ball $\mathcal{B}(s;\varepsilon)$ of s. Then $||I - \omega D\Gamma(0)^{-1}\Gamma|| < 1$ for a sufficiently small ω thus (2.73) converges as long as $s_0 \in \mathcal{B}(s;\varepsilon)$.

We remark that if there are no bound states i.e. $r = 0$, then

$$D\Gamma_0(0)p(x) = \frac{1}{2}p(\frac{x}{2}),$$

which is an isometric isomorphism. A numerical example is given in Section 2.6.

2.4 Darboux transformation

In Section 2.3 we discussed the ZSSP with bound states by considering the jump condition. From a numerical point of view, this approach is a costly iteration method since the rate of convergence is quite slow in general. Moreover it is not easy to extend to a semi-infinite problem. On the other hand, the numerical scheme for the ZSSP without bound states, for example the layer stripping method, is very fast and accurate. See Section 2.6 for numerical details. Thus, it is natural to consider a transformation $\mathfrak{D}$ such that

$$\begin{aligned} \mathfrak{D}: \quad \mathcal{S}_r &\to \mathcal{S}_0 \\ s &\mapsto s^{[0]}. \end{aligned}$$

Here

$$\mathcal{S}_r = \{s(x) \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) : |a(\zeta)| > 0 \text{ for } \zeta \in \mathbb{R}\}, \tag{2.74}$$

that is, $\mathcal{S}_r$ is a class of coefficients of the ZSSP allowed to have bound states. Similarly to $\mathcal{S}_0$ defined in (2.42) $s \in \mathcal{S}_r$ can be extended by 0 on the left half line. We drop the subscript r from $\mathcal{S}_r$ through this section.

Under this transformation to be described below, the scattering data might be changed. Let $L^{[0]}$ be the left reflection coefficient for $s^{[0]}$. Since $s^{[0]} \in \mathcal{S}_0$, $L^{[0]} \in H^2_+$ is the only scattering data. Thus $s^{[0]}$ can be recovered efficiently by the methods discussed in Section 2.2 and 2.3, and finally s can be restored by the inverse transformation of $\mathfrak{D}$. $\mathfrak{D}$ is a Darboux kind of transformation which was first investigated in the Schrödinger equation ([22, 18]). For the ZSSP, Lin showed how scattering data and coefficients are changed if a bound state is added or removed under the transformation in [41], see also [30]. The approach which we present here, is different than his work. We construct a $2N \times 2N$ system of equations which govern the relation of s and $s^{[0]}$. Moreover, in this procedure we do not need the normalizing constants $C_{r,j}$. It seems a contradiction to the general fact that a coefficient can be uniquely determined from its scattering data including bound state information. However, this condition can be weakened if a coefficient has a support restriction[2], that is, our method is derived based on a support assumption.

Consider $s \in \mathcal{S}$ and the corresponding scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$. Then $L(\zeta)$ is a meromorphic function in $\mathbb{C}^+$, and it has a pole at ζ_n. First, we seek a function $d(\zeta)$ such that $d(\zeta)L(\zeta)$ is a reflection coefficient for a coefficient of the ZSSP without bound states. One of the natural choices for $d(\zeta)$ is a simple product of $\zeta - \zeta_n$ to remove the poles. Assuming that the orders of poles are one,

$$L^{[0]}(\zeta) = d(\zeta)L(\zeta), \quad d(\zeta) = \prod_{n=1}^{N}(-i\zeta + i\zeta_n), \tag{2.75}$$

is analytic in the upper half plane. However, $L^{[0]}(\zeta)$ may not be in the Hardy space unless $L(\zeta) = O(\frac{1}{\zeta^{N+1}})$. Nevertheless, (2.75) gives a motivation for the structure of $\mathfrak{D}$.

Let

$$D(x,\zeta) := \prod_{n=1}^{N}(-i\zeta + iD_n(x)), \tag{2.76}$$

for

$$D_n(x) = \begin{pmatrix} f_n^*(x) & g_n(x) \\ -g_n^*(x) & f_n(x) \end{pmatrix}, \tag{2.77}$$

[2]See Section 2.5 for the uniqueness without bound state data

and

$$\phi^{[0]}(x,\zeta) := D(x,\zeta)\phi(x,\zeta), \tag{2.78}$$

where ϕ is the solution to the ZSSP, (2.1) with S. Here the product of matrices $\prod_{n=1}^{N} D_n$ is defined by $D_1 D_2 \cdots D_N$. We assume that $\phi^{[0]}$ solves

$$\frac{\partial \phi^{[0]}}{\partial x}(x,\zeta) = i\zeta\Lambda\phi^{[0]}(x,\zeta) + \Lambda S^{[0]}(x)\phi^{[0]}(x,\zeta), \tag{2.79}$$

for some $S^{[0]}(x) = \begin{pmatrix} 0 & {s^{[0]}}^*(x) \\ s^{[0]}(x) & 0 \end{pmatrix}$. Substitution of (2.78) into (2.79) and (2.1) give

$$\frac{\partial D}{\partial x} = -i\zeta[D,\Lambda] + \Lambda S^{[0]} D - D\Lambda S, \tag{2.80}$$

where $[\cdot,\cdot]$ denotes the usual commutator, i.e. $[D,\Lambda] = D\Lambda - \Lambda D$. We introduce $A_j(x)$ which is the coefficient of $(-i\zeta)^j$ in (2.76). That is,

$$\prod_{n=1}^{N}(-i\zeta + iD_n(x)) = \sum_{j=0}^{N} A_j(x) i^{N-j}(-i\zeta)^j. \tag{2.81}$$

We rewrite (2.80) in terms of A_j by aid of (2.81),

$$\begin{aligned} & \sum_{j=0}^{N} \frac{dA_j}{dx}\ i^{N-j}(-i\zeta)^j \\ = \quad & \sum_{j=0}^{N}[A_j,\Lambda] i^{N-j}(-i\zeta)^{j+1} + (\Lambda S^{[0]} A_j - A_j \Lambda S) i^{N-j}(-i\zeta)^j. \end{aligned}$$

Comparing with the coefficients of $(-i\zeta)^j$ yields

$$\frac{dA_j}{dx} = [A_{j-1},\Lambda i] + \Lambda S^{[0]} A_j - A_j\Lambda S, \quad j = 0,1,\cdots,N, \tag{2.82}$$

where $A_{-1} = A_N = I$.

For given S, this system of equations can be understood as a $2N \times 2N$ nonlinear first order system of equations and one algebraic equation for $\{A_j\}_{j=0}^{N-1}$ and $S^{[0]}$. That is, for $j = 0,1,\cdots,N-1$

$$\frac{dA_j}{dx} = [A_{j-1},\Lambda i] - [A_{N-1},\Lambda i]A_j + [\Lambda S, A_j], \tag{2.83}$$

$$S^{[0]} = S - \Lambda[A_{N-1},\Lambda i]. \tag{2.84}$$

Equivalently, for given $S^{[0]}$

$$\begin{aligned} \frac{dA_j}{dx} &= [A_{j-1}, \Lambda i] - A_j[A_{N-1}, \Lambda i] + [\Lambda S^{[0]}, A_j], && (2.85) \\ S &= S^{[0]} + \Lambda[A_{N-1}, \Lambda i]. && (2.86) \end{aligned}$$

In order to show the uniqueness and existence of $A_j, s^{[0]}$ (or A_j, s if $s^{[0]}$ is given), we define a Banach space $\mathcal{A}$ as

$$\{\mathbf{A} = (A_0, A_1, \cdots, A_{N-1}) : A_j = \begin{pmatrix} q_j^* & r_j \\ -r_j^* & q_j \end{pmatrix}, q_j, r_j \in L^\infty(\mathbb{R})\},$$

with $||\mathbf{A}|| := \max_j |||A_j|||_\infty$. Recall that $|A_j|$ is the matrix L^2 norm, which is $\sqrt{|q_j|^2 + |r_j|^2}$ in this case. Note that the matrix A_j has the same form as D_n since the structure of D_n, (2.77) is closed under addition and multiplication.

Lemma 2.4.1. *Suppose that $s \in L^1(\mathbb{R})$. Then (2.83) is uniquely solvable in $\mathcal{A}$ with a initial condition*

$$\mathbf{A}(x_0) = \mathbf{A}^0 = (A_1^0, A_1^0, \cdots, A_{N-1}^0), \quad |x_0| < \infty. \tag{2.87}$$

Similarly, (2.85) has a unique solution in $\mathcal{A}$ provided $s^{[0]} \in L^1(\mathbb{R})$.

Proof. We show the uniqueness and existence of (2.83) only. The similar argument may adopt to (2.85). We introduce an operator $\mathfrak{E}$ on $\mathcal{A}$ for fixed $x_0 \in \mathbb{R}$ as following.

$$(\mathfrak{E}\mathbf{A})_j(x) := \int_{x_0}^{x} [A_{j-1}(y), \Lambda i] - [A_{N-1}(y), \Lambda i]A_j(y) + [\Lambda S(y), A_j(y)]dy + A_j^0,$$

for $j = 0, 1, \cdots, N-1$. We rewrite (2.83) with the initial condition (2.87) as

$$\mathbf{A} = \mathfrak{E}\mathbf{A},$$

where, $\mathfrak{E}\mathbf{A} = ((\mathfrak{E}\mathbf{A})_0, \cdots, (\mathfrak{E}\mathbf{A})_{N-1})$. Thus, we have to show the operator $\mathfrak{E}$ has a unique fixed point in $\mathcal{A}$ for the uniqueness and existence.

Let $M := 2||\mathbf{A}^0||$. Define a Banach space $\mathcal{A}_{M,h} := \{\mathbf{A}\chi_{[x_0-h, x_0+h]} : \mathbf{A} \in \mathcal{A}, ||\mathbf{A}|| \leq M\}$ for sufficiently small h which is determined later. We claim that $\mathfrak{E}$ is a contraction mapping on

$\mathcal{A}_{M,h}$. Indeed, for $\mathbf{A} \in \mathcal{A}_{M,h}$

$$\begin{aligned}
|(\mathfrak{E}\mathbf{A})_j| &\leq \int_{x_0}^{x} |[A_{j-1}, \Lambda i]| + |[A_{N-1}, \Lambda i]A_j| + |[\Lambda S, A_j]||dy| + M/2 \\
&\leq \int_{x_0}^{x} (2M + 2M^2 + 2|s|M)|dy| + M/2 \\
&= 2M \int_{x_0}^{x} (1 + M + |s|)|dy| + M/2.
\end{aligned}$$

Since $s \in L^1(\mathbb{R})$, there exists $h > 0$ such that for any $z \in \mathbb{R}$,

$$\int_{z}^{z+h} (1 + 2M + |s|)|dy| < \frac{1}{4}.$$

For this h, $\mathfrak{E}(\mathcal{A}_{M,h}) \subset \mathcal{A}_{M,h}$.

For $\mathbf{A}^1, \mathbf{A}^2 \in \mathcal{A}_{M,h}$, let $\mathbf{E} = \mathbf{A}^1 - \mathbf{A}^2$. Then

$$\begin{aligned}
|(\mathfrak{E}\mathbf{A}^1)j - (\mathfrak{E}\mathbf{A}^2)_j| &\leq \int_{x_0}^{x} |[E_{j-1}, \Lambda i]| + |[A^1_{N-1}, \Lambda i], E_j| + |[E_{N-1}, \Lambda i], A^2_j| + |[\Lambda S, E_j]||dy| \\
&\leq \int_{x_0}^{x} 2|E_{j-1}| + 2|A^1_{N-1}||E_j| + 2|E_{N-1}||A^2_j| + 2|s||E_j||dy| \\
&\leq 2||\mathbf{E}|| \int_{x_0}^{x} (1 + 2M + |s|)|dy| < \frac{1}{2}||\mathbf{E}||.
\end{aligned}$$

This shows that $\mathfrak{E}$ is a contraction mapping on $\mathcal{A}_{M,h}$, thus we have the local uniqueness and existence for the system (2.83). Suppose that (x_1, x_2) is the maximal interval of existence, and one of end points is finite, say $x_2 < \infty$. For the global existence, we claim that the solution $\mathbf{A}$ to (2.83) is bounded at $x = x_2^-$. Given this, we can solve (2.83) with a new initial condition $\mathbf{A}(x_2)$. Thus, (x_1, x_2) is not the maximal interval, so x_2 should be infinite. Similarly, x_1 should be also infinite.

Now we verify the claim. The structure of A_j gives $|A_j|^2 I = A_j A_j^\dagger$. Since $\Lambda S A_j A_j^\dagger + A_j(\Lambda S A_j)^\dagger = 0$, from (2.82)

$$\frac{d|A_j|^2}{dx} I = [A_{j-1}, \Lambda i]A_j^\dagger + A_j[-\Lambda i, A_{j-1}^\dagger],$$

or

$$\begin{aligned}
|A_j|^2 &\leq \int_{x_0}^{x} 4|A_{j-1}||A_j|dy + |A_j^0|^2 \\
\sum_{p=0}^{N-1} |A_j|^2 &\leq 4\int_{x_0}^{x} \sum_{j=0}^{N-1} |A_{j-1}||A_j|dy + \sum_{j=0}^{N-1} |A_j^0|^2 \\
&\leq 4\int_{x_0}^{x} \sum_{j=0}^{N-1} |A_j|^2 dy + \sum_{j=0}^{N-1} |A_j^0|^2.
\end{aligned}$$

Gronwall's inequality yields

$$\sum_{j=0}^{N-1} |A_j|^2 \leq \sum_{j=0}^{N-1} |A_j^0|^2 e^{4|x-x_0|}.$$

Thus, $||\mathbf{A}(x_2)||$ is bounded if x_2 is finite. $\square$

Corollary 2.4.2. *Suppose that $s(x) = 0$ in (2.83). Then every constant function $\mathbf{A} \in \mathcal{A}$ whose components are a diagonal matrix is an invariant set.*

Now we have to find boundary conditions for (2.83). Suppose that ϕ satisfies the boundary condition (2.33). Then $\phi^{[0]}$ should satisfy

$$\phi^{[0]}(x) = \prod_{n=1}^{N}(-i\zeta + iD_n(x)) \begin{pmatrix} e^{i\zeta x} \\ L(\zeta)e^{-i\zeta x} \end{pmatrix}, \quad x < 0,$$

and

$$\begin{aligned}
(\prod_{n=1}^{N}(-i\zeta + iD_n(x)))_{2,2} &= \prod_{n=1}^{N}(-i\zeta + i\zeta_n), \quad x < 0, \\
(\prod_{n=1}^{N}(-i\zeta + iD_n(x)))_{2,1} &= 0, \quad x < 0,
\end{aligned}$$

in order to remove the poles of $L(\zeta)$ as mentioned earlier (see (2.75)). From Corollary 2.4.2, we have boundary conditions for (2.83) as

$$f_n(0) = \zeta_n, \quad g_n(0) = 0, \quad n = 1, \cdots, N. \tag{2.88}$$

In this case, however, $L^{[0]}(\zeta)$ may not be a square integrable function. Besides, the first component of $\phi^{[0]}$ is not $e^{i\zeta x}$ for $x < 0$. Thus we define a Darboux matrix $\mathbb{D}$ as

$$\mathbb{D} = D(x,\zeta)\phi(x,\zeta) \prod_{n=1}^{N} \frac{1}{-i\zeta + i\zeta_n^*}.$$

Then under the action of $\mathbb{D}$,

$$L^{[0]}(\zeta) = L(\zeta)\prod_{n=1}^{N}\frac{\zeta-\zeta_n}{\zeta-\zeta_n^*}, \tag{2.89}$$

which is in H_+^2 as desired.

The Darboux transformation $\mathfrak{D}$ which we defined can remove N bound states by solving a $2N\times 2N$ system of equations, but it can be considered as N composition of $\mathfrak{D}^{[n]}$ which is a transformation to remove n-th bound state. Formally, we write

$$\mathfrak{D} = \mathfrak{D}^{[1]}\mathfrak{D}^{[1]}\cdots\mathfrak{D}^{[N]},$$

where, $\mathfrak{D}^{[n]}(s^{[n]}) = s^{[n-1]}$ is defined by

$$\begin{aligned}\frac{dD_n}{dx} &= -[D_n,\Lambda i]D_n + [\Lambda S^{[n]}, D_n], \quad D_n(0) = \begin{pmatrix}\zeta_n^* & 0\\ 0 & \zeta_n\end{pmatrix} && (2.90)\\ S^{[n-1]} &= S^{[n]} - \Lambda[D_n,\Lambda i]. && (2.91)\end{aligned}$$

By considering the inverse of $\mathfrak{D}^{[n]}$, one can generate $S^{[n]}$ which has one more bound state at $\zeta=\zeta_n$ from $S^{[n-1]}$. Obviously, the inverse of $\mathfrak{D}^{[n]}$ is defined through

$$\begin{aligned}\frac{dD_n}{dx} &= -D_n[D_n,\Lambda i] + [\Lambda S^{[n-1]}, D_n], \quad D_n(0) = \begin{pmatrix}\zeta_n^* & 0\\ 0 & \zeta_n\end{pmatrix} && (2.92)\\ S^{[n]} &= S^{[n-1]} + \Lambda[D_n,\Lambda i]. && (2.93)\end{aligned}$$

We rewrite (2.90) componentwise from the structure of D_n given in (2.77) as follows.

$$\begin{aligned}\frac{df_n}{dx} &= 2|g_n|^2 i - s g_n + s^* g_n^*, \quad f_n(0)=\zeta_n, && (2.94)\\ \frac{dg_n}{dx} &= 2f_n g_n i + f_n s^* - f_n^* s^*, \quad g_n(0)=0. && (2.95)\end{aligned}$$

Note that $|f_n|^2+|g_n|^2 = |\zeta_n|^2$ for all x, because $d(|f_n|^2+|g_n|^2)/dx = 0$. Moreover, (2.94) says that the real part of f_n remains a constant. We separate f_n, g_n into real parts and imaginary parts $f_n = f_R + f_I i, g_n = g_R + g_I i, s = s_R + s_I i$. It follows that

$$\begin{aligned}\frac{df_I}{dx} &= 2g_R^2 + 2g_I^2 - 2s_R g_I - 2s_I g_R, && (2.96)\\ \frac{dg_R}{dx} &= -2f_I g_R - 2f_R g_I + 2s_I f_I, && (2.97)\\ \frac{dg_I}{dx} &= 2f_R g_R - 2f_I g_I + 2s_R f_I. && (2.98)\end{aligned}$$

Suppose that s is compactly supported on $[0, X]$. Then $f_I^2(X)+g_R^2(X)+g_I^2(X) = (\mathrm{Im}\zeta_0)^2 =: k_0^2$ for $k_0 > 0$[3]. If $f_I(X) = \pm k_0$, then $g_n = 0$ for $x \geq X$ from Corollary 2.4.2. Now assume that $f_I(X) \neq \pm k_0$. It is not difficult to show that (f_I, g_R, g_I) converges to $(k_0, 0, 0)$ exponentially. That is, $(k_0, 0, 0)$ is an asymptotically stable point for

$$\frac{df_I}{dx} = 2g_R^2 + 2g_I^2, \tag{2.99}$$
$$\frac{dg_R}{dx} = -2f_I g_R - 2f_R g_I, \tag{2.100}$$
$$\frac{dg_I}{dx} = 2f_R g_R - 2f_I g_I. \tag{2.101}$$

We rewrite above two system of equations, (2.96)-(2.98) and (2.99)-(2.101) as

$$\frac{dy}{dx} = G(y) + H(s, y), \tag{2.102}$$
$$\frac{dy}{dx} = G(y), \tag{2.103}$$

respectively, where $y = (f_I, g_R, g_I)^T$.

Now we consider the case of $s \in L^1(\mathbb{R}^+)$. We can linearize (2.102) at $(k_0, 0, 0)$ as

$$\frac{dy}{dx} = Ay + (G(y) - Ay) + H(s, y), \tag{2.104}$$

where $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2k_0 & -2f_R \\ 0 & 2f_R & -2k_0 \end{pmatrix}$.

Since $s \in L^1(\mathbb{R}^+)$ and $|H(s, y)| \leq 2|s||y|$, the nonlinear term

$$(G(y) - Ay) + H(s, y) = o(y)$$

uniformly for $x \in [X, \infty)$ with sufficiently large X. This shows that (2.104) is exponentially asymptotically stable at $(k_0, 0, 0)^T$ from the well known ODE theory (see e.g [31] or [42]). It follows from (2.91) that $s^{[n-1]} \in L^1(\mathbb{R}^+)$(or $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$) if $s^{[n]} \in L^1(\mathbb{R}^+)$(or $L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$) since g_n vanishes exponentially or is zero on $[X, \infty)$ for some $X > 0$.

We remark that Lemma 2.4.1 can be shown by the uniqueness and existence of (2.90).

[3] Recall that $\mathrm{Im}\zeta_0 > 0$

Theorem 2.4.3. *Suppose that $s \in \mathcal{S}_r$ with the scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$ in the ZSSP. Then*

$$\begin{aligned} \mathfrak{D}: \quad \mathcal{S}_r &\to \mathcal{S}_0 \\ s &\mapsto s^{[0]} \end{aligned}$$

is well defined by (2.83),(2.84) and (2.88), and it is an injective map. The corresponding left reflection coefficient $L^{[0]}(\zeta)$ is given by (2.89).

Conversely, for given $\{\zeta_n\}_{n=1}^N \subset \mathbb{C}^+$, $s^{[0]} \in \mathcal{S}_0$, and $L^{[0]}$ one can construct a unique $s \in \mathcal{S}_r$ which generates the left reflection coefficient $L = L^{[0]} \prod_{n=1}^{N} \frac{\zeta - \zeta_n^*}{\zeta - \zeta_n}$ and has bound states at $\{\zeta_n\}_{n=1}^N$ via (2.85), (2.86) and (2.88).

With Theorem 2.2.5, we have the following uniqueness for the ZSSP as a corollary of Theorem 2.4.3.

Corollary 2.4.4. *Suppose that a coefficient $s(x)$ in the ZSSP is in the class $\mathcal{S}_r$. Then $s(x)$ is uniquely determined from its scattering data.*

2.5 Uniqueness without bound state information

We showed the left reflection coefficient uniquely determines $s(x) \in \mathcal{S}_0$ in Section 2.2. In case of $s(x) \in \mathcal{S}_r$, we developed two methods to recover $s(x)$ from the left scattering data. In general, the reflection coefficient is not enough to reconstruct $s(x)$ if it has bound states. For example, the coefficient $s(x)$ given by

$$s(x) = ic^* e^{2i\zeta_0^* x} / [1 - \frac{|c|^2}{(\zeta_0 - \zeta_0^*)^2} e^{2i(\zeta_0 - \zeta_0^*)x}] \tag{2.105}$$

has one bound states $\{\zeta_0, c\}$ but the right reflection is zero ([5]). Obviously, the zero coefficient has zero reflection coefficient. Similarly to the Schrödinger scattering problem as in [7, 44], however, a support restriction on $s(x)$ might be such that a reflection coefficient only determines $s(x)$.

Theorem 2.5.1. *Suppose that $s(x) \in \mathcal{S}_r$. Then $L(\zeta)$ uniquely determines $s(x)$. Similarly $R(\zeta)$ determines $s(x)$ in the case of support in the left half line, i.e. $s(x) \in \mathcal{S}_l$. Here,*

$$\mathcal{S}_l = \{s(x) \in L^1(\mathbb{R}^-) \cap L^2(\mathbb{R}^-) : |a(\zeta)| > 0 \text{ for } \zeta \in \mathbb{R}\}.$$

Proof. First we claim that if $s \in \mathcal{S}_r$

$$\widehat{L}(t) = \sum_{n=1}^{N} iC_{r,n} e^{-i\zeta_n t}, \quad t < 0. \tag{2.106}$$

Given this, we can show that the bound states data $\{\zeta_n, C_{r,n}\}_{n=1}^N$ are uniquely determined by knowledge of $L(\zeta)$. Let

$$f(t) = -\sum_{n=1}^{N} C_{r,n} e^{i\zeta_n t}, \quad g(x) = -\sum_{m=1}^{M} D_m e^{i\xi_m t}$$

be two functions hold (2.106) for fixed $\widehat{L}(t)$. Trivially $f(t)$ and $g(t)$ can be extended to holomorphic functions. By the assumption, we have $f(z) \equiv g(z)$. Now, we need to show that $N = M$,$\zeta_j = \xi_j$, and $C_{r,j} = D_j$. For this, it is enough to show that $\{e^{\kappa_j z}\}_{n=1}^N$ is linearly independent for any N. Assume that $\kappa_i \neq \kappa_j$ for $i \neq j$. Suppose

$$c_1 e^{\kappa_1 z} + \cdots + c_N e^{\kappa_N z} = 0.$$

Then by substituting 0 in z after differentiating

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \kappa_1 & \kappa_2 & \cdots & \kappa_N \\ \cdots & \cdots & \cdots & \cdots \\ \kappa_1^{N-1} & \kappa_2^{N-1} & \cdots & \kappa_N^{N-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

Since the Vandermonde determinant is given by $\prod_{i<j}(\kappa_j - \kappa_i)$, which is nonzero by the assumption, $c_j = 0$ for all j. Thus we have desired result.

Now we verify the claim (2.106). From (2.5), $N_x(x,y) + \Gamma N_z(x,y) = 0$ for $y < x < 0$ since $s(x) = 0$ for $x < 0$. Then it follows from the boundary condition (2.7) that

$$N(x,y) = \begin{pmatrix} G(x-y) \\ 0 \end{pmatrix}, \quad y < x < 0$$

for some function G. Substituting this representation into the GLM equation (2.15), there follows

$$\begin{pmatrix} G(x-y) \\ 0 \end{pmatrix} + M(x+y)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 0 \\ G^*(x-y) \end{pmatrix} M(z+y)dz = 0.$$

Immediately $G(x-y) = 0$ from the first row, and it follows from the second row that $M(x+y) = 0$ for $y < x < 0$. Thus, we have (2.106) from (2.16). □

Note that (2.106) can be shown by several methods. For example, the asymptotic behavior of $L(\zeta)$ in $\mathbb{C}^+$ and the residue theorem may be used for the proof. Also an integral representation for $N(x,y)$ derived in Proposition 1 in [52] can give another proof.

2.6 Numerical Algorithm

2.6.1 Born approximation

Let $\phi(x,\zeta) = \begin{pmatrix} A(x,\zeta) \\ B(x,\zeta) \end{pmatrix}$. Then (2.1) can be rewritten

$$A_x = i\zeta A + s^* B, \tag{2.107}$$

$$B_x = -i\zeta B - sA. \tag{2.108}$$

Assume that $s(x)$ is supported on $[0, X]$, and there are no bound states. Then the scattering data consists only of a reflection coefficient $L(\zeta)$, which is given by $B(0,\zeta)$ if the boundary conditions

$$A(0,\zeta) = 1, \quad B(X,\zeta) = 0 \tag{2.109}$$

are assumed. For a parameter ε, let $s = \sum_{j=1}^{\infty} \varepsilon^j s_j$. Then the corresponding solutions are in the form of

$$\begin{pmatrix} A \\ B \end{pmatrix} = \sum_{j=0}^{\infty} \varepsilon^j \begin{pmatrix} A_j \\ B_j \end{pmatrix}.$$

Substituting these representation in (2.107) and (2.108), there follows

$$\begin{aligned} A_{0,x} + \varepsilon A_{1,x} + O(\varepsilon^2) &= i\zeta A_0 + \varepsilon(i\zeta A_1 + s_1^* B_0) + O(\varepsilon^2), \\ B_{0,x} + \varepsilon B_{1,x} + O(\varepsilon^2) &= -i\zeta B_0 + \varepsilon(-i\zeta B_1 - s_1 A_0) + O(\varepsilon^2), \end{aligned}$$

or, with the boundary conditions (2.109)

$$\begin{aligned}
A_{0,x} &= i\zeta A_0, \quad A_0(0) = 1, \\
B_{0,x} &= -i\zeta B_0, \quad B_0(X) = 0, \\
A_{1,x} &= i\zeta A_1 + s_1^* B_0, \quad A_1(0) = 0, \\
B_{1,x} &= -i\zeta B_1 - s_1 A_0, \quad B_1(X) = 0.
\end{aligned}$$

The first and the last equations yield

$$B_1(x,\zeta)e^{i\zeta x} = \int_X^x -s_1(z)e^{2i\zeta z}dz,$$

or

$$B_1(0,\zeta) = \int_0^X s_1(z)e^{2i\zeta z}dz. \tag{2.110}$$

The inverse Fourier inverse transformation gives

$$s_1(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} B_1(0,\zeta)e^{-2i\zeta x}d\zeta. \tag{2.111}$$

Suppose that ε is sufficiently small. Then $s_1(x) \sim s(x)$ and $B_1(0,\zeta) \sim L(\zeta)$, thus

$$s(x) \sim \frac{1}{\pi}\int_{-\infty}^{\infty} L(\zeta)e^{-2i\zeta x}d\zeta.$$

This formula which is known as the Born approximation can successfully recover $s(x)$ as long as $s(x)$ is small enough, see [46] and references therein.

Another application of the Born approximation is tracking a discontinuity of $s(x)$. Suppose that $s(x)$ is continuously differentiable except at the origin, that is $s(0^+) \neq 0$. Then integration by parts on (2.110) gives

$$B_1(0,\zeta) = -\frac{1}{2i\zeta}s_1(0^+) - \int_0^X \frac{e^{2i\zeta z}}{2i\zeta}\frac{ds_1(z)}{dz}dz,$$

thus

$$\lim_{|\zeta|\to\infty} \zeta L(\zeta) \sim \frac{s(0)}{2}i. \tag{2.112}$$

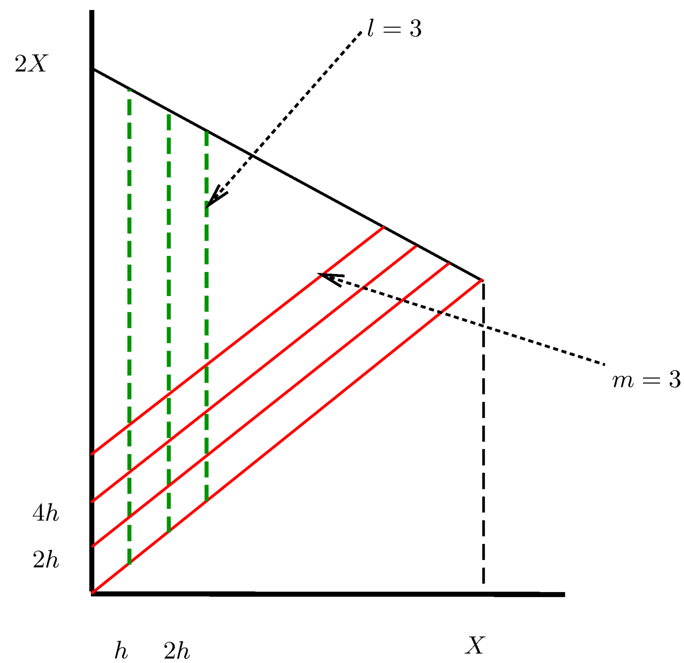

Figure 2.2 Finite difference grid I

2.6.2 Layer stripping method

Here, we derive an algorithm to solve the ZSSP based on the layer stripping method which is developed for the scalar problem in [48]. Recall that the time domain problem for the non-bound state ZSSP in case of compact support is described as

$$A_x + A_t = s^* B, \quad B_x - B_t = -sA, \qquad \text{in } \mathbf{U}_{u,X}, \tag{2.113}$$

$$A(0,t) = 0, \quad B(0,t) = \widehat{L}(t), \qquad 0 \le t \le 2X, \tag{2.114}$$

$$B(x,x) = \tfrac{1}{2} s(x), \qquad 0 \le x \le X. \tag{2.115}$$

Here, we drop the subscript $\cdot^u$.

The total derivative with respect to x along the characteristic lines $\dfrac{dt}{dx} = 1$, $\dfrac{dt}{dx} = -1$ is given by

$$\frac{d(\cdot)}{dx} = \frac{\partial(\cdot)}{\partial x} + \frac{\partial(\cdot)}{\partial t}\frac{dt}{dx}.$$

Thus, the equations (2.113) lead

$$\frac{dA}{dx} = A_x + A_t = s^* B \quad \text{for } \frac{dt}{dx} = 1, \tag{2.116}$$
$$\frac{dB}{dx} = B_x - B_t = -sA \quad \text{for } \frac{dt}{dx} = -1. \tag{2.117}$$

Now consider a finite difference grids shown in Figure 2.2. For uniform mesh size h and $2h$ for x-axis and t-axis respectively such that $Nh = X$, we have

$$(x,t) = (lh, (2m+l)h), \quad l, m = 0, 1, \cdots, N.$$

We denote that

$$s_l = s(lh), \quad A_l^m = A(lh, (2m+l)h), \quad B_l^m = B(lh, (2m+l)h).$$

Then the discrete characteristic boundary condition for (2.115) is

$$s_l = 2B_l^0, \tag{2.118}$$

in particular,

$$s_0 = 2B_0^0 = \widehat{L}(0).$$

There are several finite difference methods to approximate (2.116),(2.117). Here, we use the backward Euler scheme and the central scheme.

- *Backward Euler scheme*

  By the backward Euler scheme, the equations (2.116), (2.117) can be approximated by

$$\frac{A_l^m - A_{l-1}^m}{h} = s_l^* B_l^m, \qquad \text{for } \frac{dt}{dx} = 1,$$
$$\frac{B_l^m - B_{l-1}^{m+1}}{h} = -s_l A_l^m, \qquad \text{for } \frac{dt}{dx} = -1.$$

  In a matrix form,

$$\begin{pmatrix} 1 & -hs_l^* \\ hs_l & 1 \end{pmatrix} \begin{pmatrix} A_l^m \\ B_l^m \end{pmatrix} = \begin{pmatrix} A_{l-1}^m \\ B_{l-1}^{m+1} \end{pmatrix},$$

  or,

$$\begin{pmatrix} A_l^m \\ B_l^m \end{pmatrix} = \frac{1}{1 + h^2 |s_l|^2} \begin{pmatrix} 1 & hs_l^* \\ -hs_l & 1 \end{pmatrix} \begin{pmatrix} A_{l-1}^m \\ B_{l-1}^{m+1} \end{pmatrix}. \tag{2.119}$$

So, we can update A_l, B_l from A_{l-1}, B_{l-1} if we know s_l. On the other hand, (2.119) gives s_l together with the characteristic boundary condition (2.118). Indeed, at $m = 0$

$$B_l^0 = \frac{-hs_l A_{l-1}^0 + B_{l-1}^1}{1 + h^2|s_l|^2} \tag{2.120}$$

$$= \frac{-2hA_{l-1}^0 B_l^0 + B_{l-1}^1}{1 + 4h^2|B_l^0|^2}. \tag{2.121}$$

Suppose that h, A_{l-1}^0, and B_{l-1}^1 are known data. Then (2.121) is a cubic equation for B_l^0. For a moment, let $x = B_l^0, a = A_{l-1}^0, b = B_{l-1}^1$.

$$x = \frac{-2ahx + b}{1 + 4h^2|x|^2} =: f(x) \tag{2.122}$$

Although it might be difficult to find an analytic solution of (2.122), we can approximate the solution by a successive iteration method. Assuming x is bounded, say $|x| \le M$,

$$\begin{aligned} |f(x) - f(y)| &= |\frac{-2ahx + b}{1 + 4h^2|x|^2} - \frac{-2ahy + b}{1 + 4h^2|y|^2}| \\ &\le |4bh^2(|y|^2 - |x|^2) - 2ah(y - x) - 8ah^3(y|x|^2 - x|y|^2)| \\ &\le 4|b|h^2 2M|x - y| + 2|a|h|x - y| + 8|a|h^3M^2|x - y| \\ &\le C|x - y|. \end{aligned}$$

If the mesh size h is sufficiently small so that $C < 1$, then $f(x)$ has a unique fixed point in $|x| \le M$.

- *Central Scheme*

The equations (2.116), (2.117) may be approximated by the central scheme as follows.

$$\frac{A_l^m - A_{l-2}^m}{2h} = s_{l-1}^* B_{l-1}^m, \qquad \text{for } \frac{dt}{dx} = 1,$$

$$\frac{B_l^m - B_{l-2}^{m+1}}{2h} = -s_{l-1} A_{l-1}^m, \qquad \text{for } \frac{dt}{dx} = -1,$$

or on the l_{th} vertical line,

$$\begin{aligned} A_l^m &= 2hs_{l-1}^* B_{l-1}^m + A_{l-2}^m \\ B_l^m &= -2hs^{l-1} A_{l-1}^m + B_{l-2}^{m+2}. \end{aligned}$$

From this equations, A_l, B_l can be computed from $A_{l-1}, B_{l-1}, A_{l-2}, B_{l-2}$, and s_{l-1}. Then s_l is given by (2.118). Note that we do not need any iterations in this method in contrast with the backward Euler scheme.

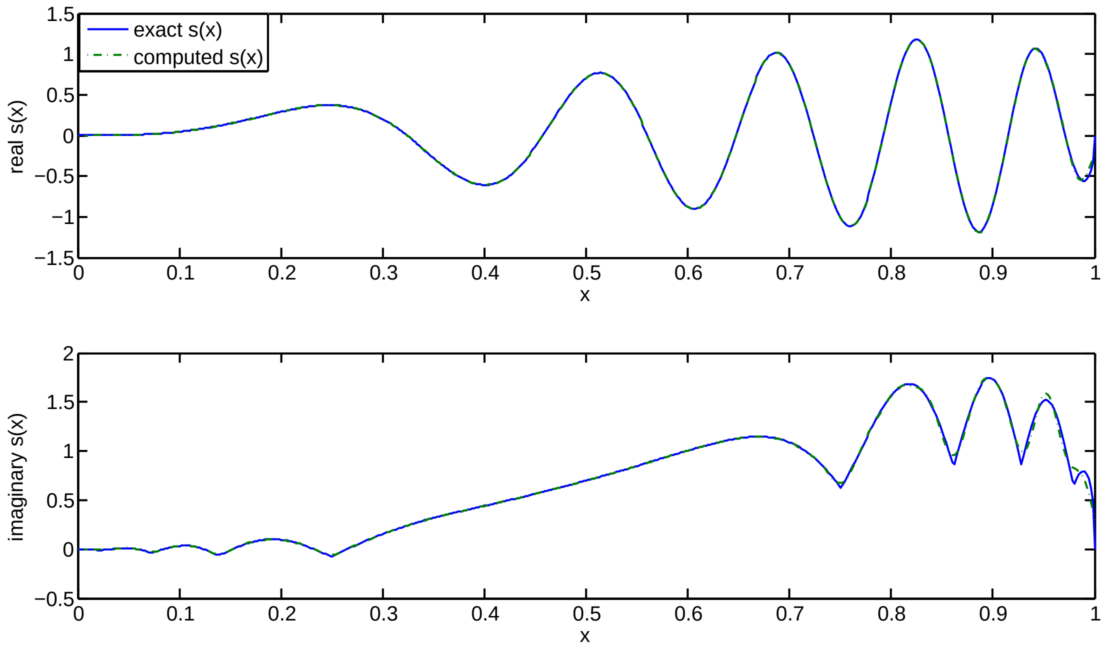

Figure 2.3 Backward Euler scheme

Table 2.1 Residuals in backward Euler scheme

step of iteration	$n = 1$	$n = 3$	$n = 5$
$\mathbf{x = 0.2}$	0.006726526548038	0.000000000000309	0
$\mathbf{x = 0.6}$	0.019308593719365	0.000000003146410	0
$\mathbf{x = 1}$	0.115419150737459	0.000000477188676	0.000000000001973

Figure 2.3 and 2.4 show numerical examples of $s(x)$ reconstructed on $[0, 1]$ from $L(\zeta)$ by the backward Euler scheme and the central scheme respectively. The sampled left reflection coefficient $L(\zeta_j)$ for $\zeta_j = j\Delta\zeta,\ j = -M, \cdots, M$ is generated by an ODE solver. In computing $\widehat{L}(t_k)$ for $t_k = 2\Delta x,\ k = 0, \cdots, N$ such that $t_N = 2$, we used the fast Fourier transformation algorithm with padding the $L(\zeta_j)$ by zero to an interval of $[-(M + M')\Delta\zeta, (M + M')\Delta\zeta]$.

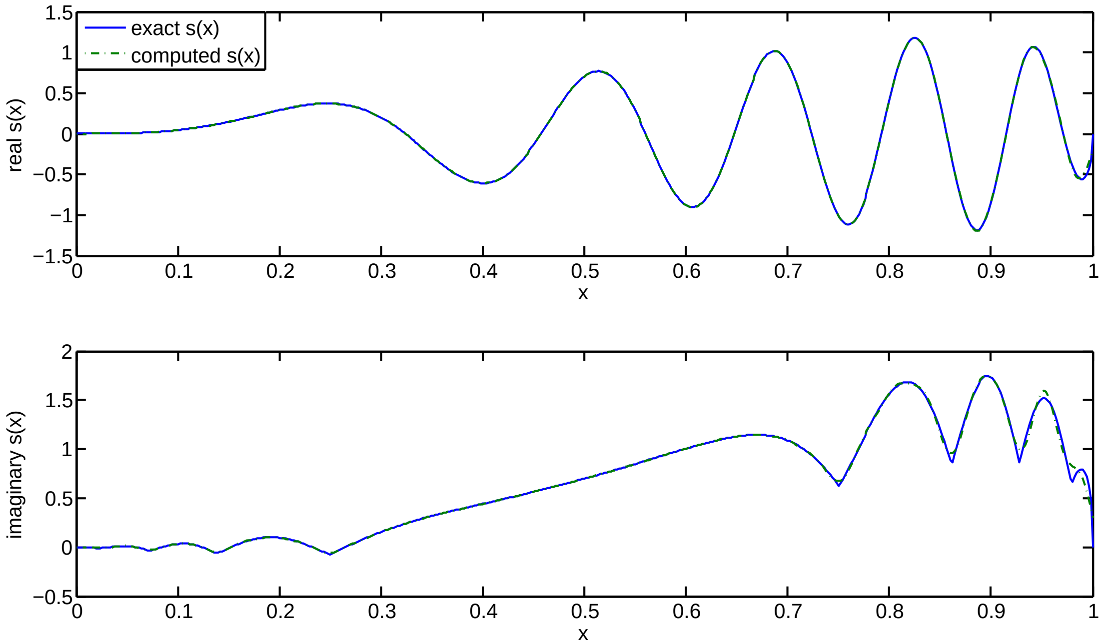

Figure 2.4 Central scheme

The coefficient $s(x_k)$ is reconstructed at $x_k = k\Delta x,\ k = 0, \cdots, N$. In these examples, we use $\Delta\zeta = 0.01, M = 10000, M' = 10^6$, and $\Delta x = 0.002$.

Table 2.1 shows residual at each iteration step to solve (2.122) in the backward Euler scheme. We iterated 10 times to solve the fixed point problem at each point.

2.6.3 Modified Newton's method

Here, we give numerical examples of the ZSSP via the following modified Newton's method introduced in Section 2.3.

$$s_{n+1} = (I - \omega D\Gamma(0)^{-1}\Gamma)s_n + \omega D\Gamma(0)^{-1}\widehat{L}. \tag{2.123}$$

$D\Gamma(0)$ is defined in (2.58), that is, for a given transmission coefficient $T(\zeta)$,

$$D\Gamma(0)s(x) = \frac{1}{2}s(\frac{x}{2}) + \frac{1}{2}\int_x^{2X} s(\frac{z}{2})r(z-2x)dz,$$

where $r(x) = \widehat{T-1}(-x)$. Recall that $s(x)$ allows bound states in this method. In Theorem 2.3.4, we showed that $D\Gamma(0)^{-1}$ exists, which is given by a Volterra integral equation of the second kind. That is, if $D\Gamma(0)^{-1}p(x) = s(x)$ then s is a solution to the following equation.

$$p(x) = \frac{1}{2}s(\frac{x}{2}) + \frac{1}{2}\int_x^{2X} s(\frac{z}{2})r(z-2x)dz. \tag{2.124}$$

Recall the definition of $\Phi(s, f, g)$,

$$\begin{aligned}\Phi: \quad L^2(0,X)\times L^2(0,2X)\times L^2(0,2X) &\to L^2(0,X)\\ (s(x), f(t), g(t)) &\mapsto B(x,x)\end{aligned}$$

where $B(x,t)$ solves

$$\begin{aligned}A_x + A_t = s^*B, \quad B_x - B_t = -sA, &\qquad \text{in } \mathbf{U}_{u,X}\\ A(0,t) = f(t), \quad B(0,t) = g(t), &\qquad 0 \le t \le 2X.\end{aligned}$$

For fixed s, f, we may define the inverse map $\Phi^{-1}_{s,f}$ of $\Phi(s, f, \cdot\ )$. That is, $\Phi^{-1}_{s,f}p$ is given by $B(0,t), 0 \le t \le 2X$ for a solution to the following characteristic boundary value problem,

$$A_x + A_t = s^*B, \quad B_x - B_t = -sA, \qquad \text{in } \mathbf{U}_{u,X} \tag{2.125}$$

$$A(0,t) = f(t), \quad B(x,x) = p(x), \qquad 0 \le t \le 2X, 0 \le x \le X. \tag{2.126}$$

One can show that above system of equations has a unique solution ([27, 21]). Together with the jump condition (2.35), then $\Gamma(s)$ is defined by

$$\Gamma(s) = \Phi^{-1}_{s,0}(\frac{1}{2}s + \mathcal{R}\Phi(-\mathcal{R}s, r, 0)). \tag{2.127}$$

Recall that $\mathcal{R}$ is a reflection operator defined by $\mathcal{R}(f)(x) = f(X-x)$ for any function f.

Numerically, $\Phi(s, f, g)$ might be computed by a finite difference method, for example, we can use (2.119). Similarly, $\Phi^{-1}_{s,f}p$, or (2.125), (2.126) can be computed via a finite difference method. Consider a finite difference grid shown in Figure 2.5.

We write (2.125) as a difference equation by the backward Euler method.

$$\begin{aligned}\frac{A^n_l - A^{n-1}_{l-1}}{h} &= s^*_l B^n_l, \qquad \text{for } \frac{dt}{dx} = 1,\\ \frac{B^n_l - B^n_{l-1}}{h} &= -s_l A^n_l, \qquad \text{for } \frac{dt}{dx} = -1,\end{aligned}$$

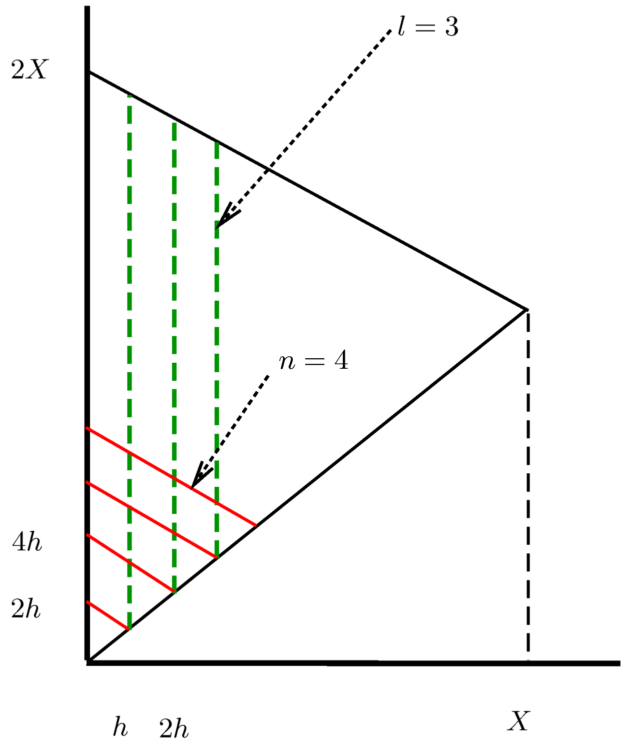

Figure 2.5 Finite difference grid II

or

$$\begin{aligned} A_l^n &= hs_l^* B_l^n + A_{l-1}^{n-1}, \\ B_{l-1}^n &= hs_l A_l^n + B_l^n. \end{aligned}$$

Here, A_l^n, B_l^n are values of A, B at $(x,t) = (lh, 2nh - lh)$ for $l, n = 0, 1, \cdots, N$. For given boundary conditions

$$A_0^n = f(2nh), \quad B_n^n = p(nh, nh),$$

the numerical solution A, B to (2.125) can be computed by the following algorithm.

Algorithm 2.6.1.

For $n = 1 : N$

$A_0^n = f(2nh)$, $B_n^n = p(nh, nh)$

For $l = n : 1$

$$A_l^n = hs_l^* B_l^n + A_{l-1}^{n-1}$$

$$B_{l-1}^n = hs_l A_l^n + B_l^n$$

End

End

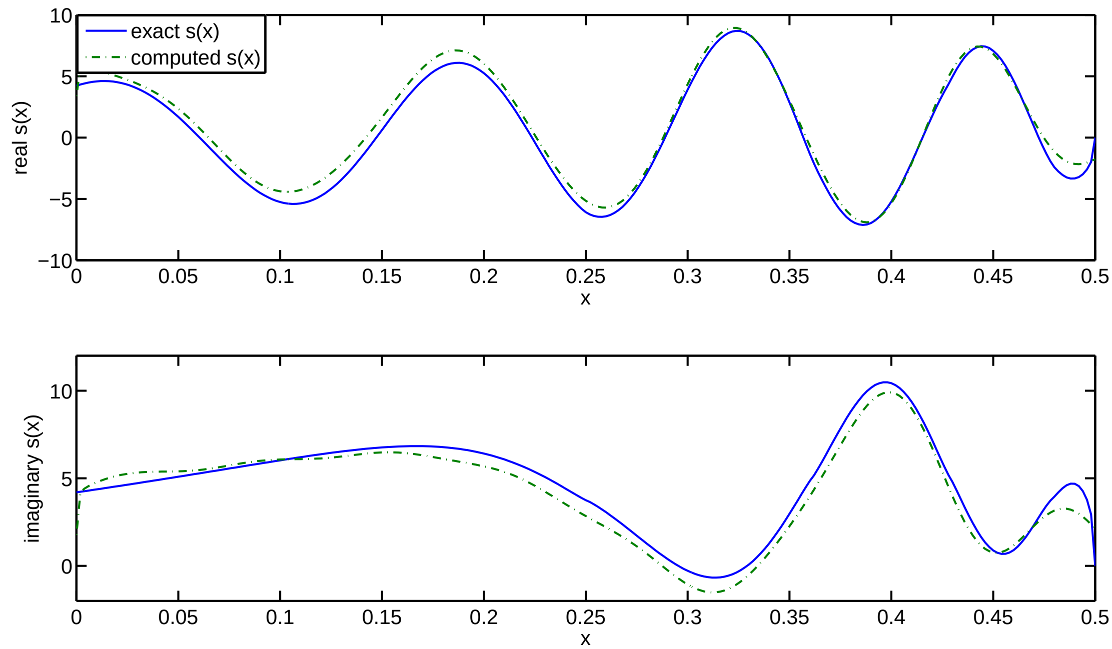

Figure 2.6 Modified Newton's method: $\omega = 1/60$, after 500 iterations

We summarize the modified Newton's method as follows.

Algorithm 2.6.2.

1. *For given scattering data* $\{L(\zeta), T(\zeta)\}$, *compute Fourier transform* $\widehat{L}(t), \widehat{T-1}(t)$ *on* $t \geq 0, t \leq 0$ *respectively.*

2. *Suppose* $s_n(t)$ *is given. Compute* $\Gamma(s_n)$ *by* (2.127). Φ *and* $\Phi_{s,0}^{-1}$ *can be computed by* (2.119) *and Algorithm 2.6.1.*

3. *Solve the Volterra integral equations* (2.124) *with* $p = \Gamma(s_n)$ *and* $p = \widehat{L}$ *respectively.*

4. *Updated* s_{n+1} *by* (2.123).

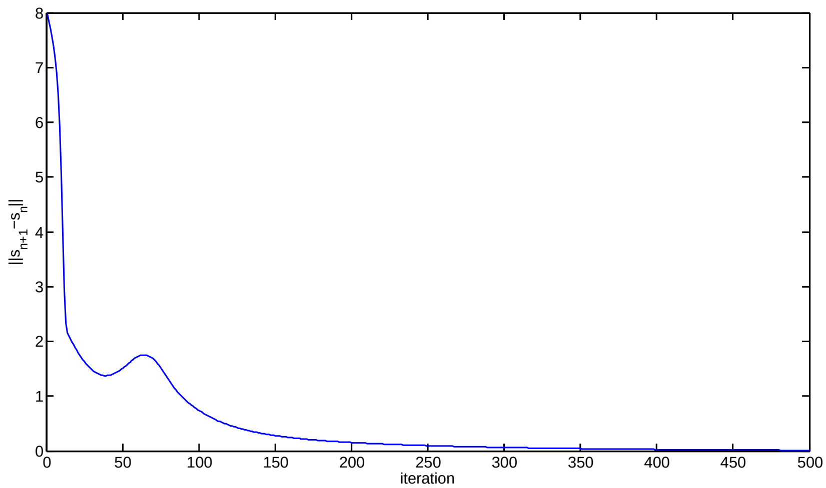

Figure 2.7 Residuals in the modified Newton's method

Figure 2.6 shows a reconstructed $s(x)$ from $L(\zeta)$ and $T(\zeta)$ which were generated by an ODE solver. $s(x)$ has at least one bound state. We set $\omega = 1/60$ and $s_0 = 0$ as an initial guess. Residual at each step is shown in Figure 2.7.

Now, suppose that $s(x)$ does not involve any bound states. Then, as pointed out at the last paragraph in Section 2.3, $D\Gamma_0(0)$ is a simple isometric isomorphism. The induced modified Newton's method is

$$s_{n+1}(x) = s_n(x) - D\Gamma_0(0)^{-1}(\Gamma(s_n)(x) - \widehat{L}(x)),$$

or

$$s_{n+1}(x) = s_n(x) - 2(\Gamma(s_n)(2x) - \widehat{L}(2x)), \tag{2.128}$$

since $D\Gamma_0(0)p(x) = \dfrac{1}{2}p(\dfrac{x}{2})$. In this scheme, we do not need damping factor ω. Moreover, $\Gamma(s_n)$ is simply given by $\Phi^{-1}_{s_n,0}(\dfrac{1}{2}s_n)$ from the properties of Φ introduced in Lemma 2.3.3. Figure 2.8 shows the numerical example of $s(x)$ computed after 10 iterations. It converges very rapidly. Residuals are given in Table 2.2.

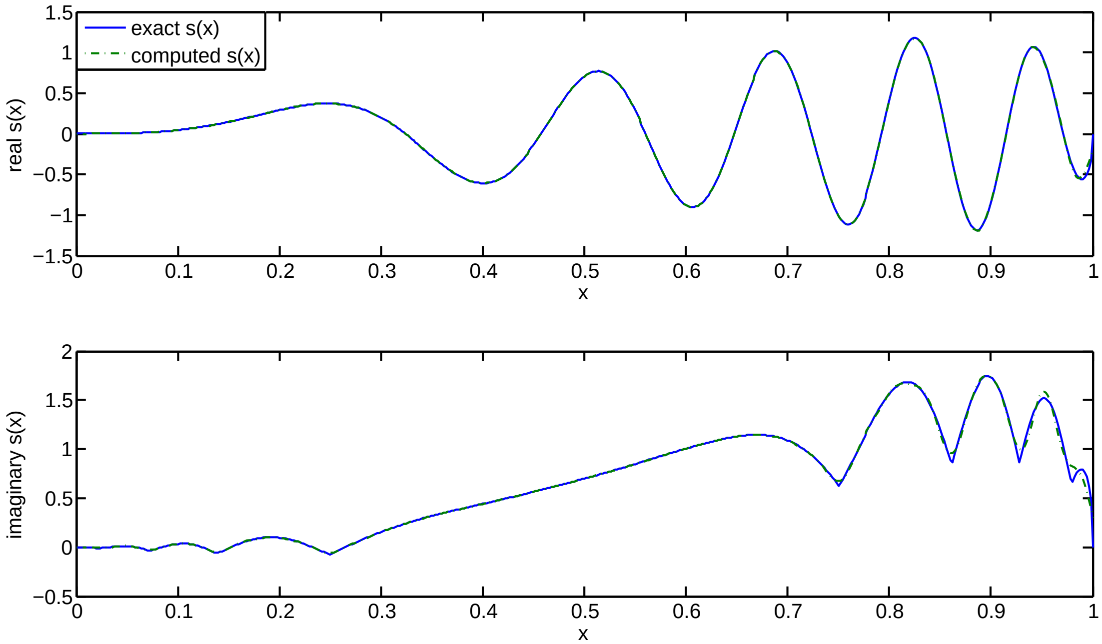

Figure 2.8 Modified Newton's method: no bound states, after 10 iterations

Table 2.2 $||s_n - s_{n-1}||$ in case of no bound states

$n = 1$	$n = 2$	$n = 3$	$n = 10$
23.787628105120287	2.217930116307225	0.388561843570568	0.000000003902530

2.6.4 Reconstruct coefficient without bound state information

In this section, we construct s without bound state information based on Theorem 2.5.1 and the Darboux transformation $\mathfrak{D}$. To extract the bound state data from $\widehat{L}$ is a very ill-posed problem if $N > 1$ (see [53] for a numerical method), even though we know they are uniquely determined. So, in this work we assume that $s(x)$ has at most one bound state.

The following algorithm reconstructs $s(x)$ from its left reflection coefficient $L(\zeta)$ provided $s \in \mathcal{S}_r$ and it has only one bound state.

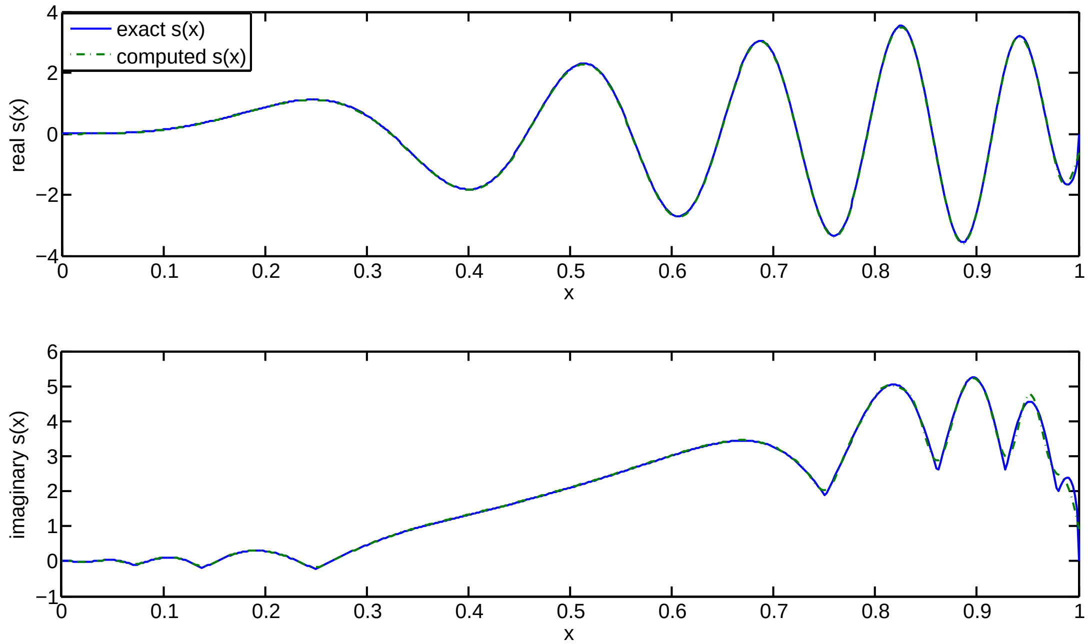

Figure 2.9 Darboux transformation without bound state information

Algorithm 2.6.3.

1. *From the given data $L(\zeta)$, compute bound state ζ_1 from*

$$\zeta_1 = \frac{d\widehat{L}(t)}{dt}\frac{i}{\widehat{L}(t)}, \quad t < 0. \tag{2.129}$$

2. *Define $L^{[0]}(\zeta) = L(\zeta)\dfrac{\zeta - \zeta_1}{\zeta - \zeta_1^*}$.*

3. *Reconstruct $s^{[0]}(x)$ from $L^{[0]}(\zeta)$.*

4. *Restore $s(x)$ by solving the following system of equations which is equivalent to (2.85).*

$$\begin{aligned} \frac{df}{dx} &= -2i|g|^2 + s^{[0]*}g^* - s^{[0]}g, \quad f(0) = \zeta_1. \\ \frac{dg}{dx} &= 2if^*g + s^{[0]*}f - s^{[0]*}f^*, \quad g(0) = 0. \end{aligned}$$

$s(x)$ is given by (2.86), or

$$s = s^{[0]} + 2g^*i. \tag{2.130}$$

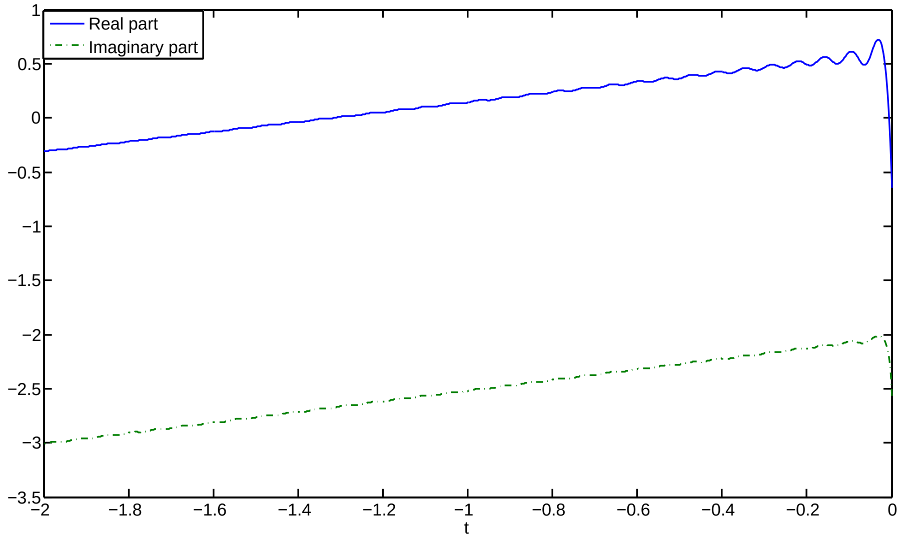

Figure 2.10 $\log \widehat{L}(t)$: slope $= -i\zeta_1$

Figure 2.9 shows a reconstructed coefficient $s(x)$ via Algorithm 2.6.3. The eigenvalue is extracted from the slope of $\log \widehat{L}(t)$ for $t < 0$. Figure 2.10 shows a shape of $\log \widehat{L}(t)$. The extracted eigenvalue is $\zeta_1 = -0.020255867465371 + 0.932450595121938i$.

We remark that $s^{[0]}(x)$ might not be compactly supported even if $s(x)$ is supported on $[0, X]$. However, from a numerical point of view, we need information of $s^{[0]}(x)$ only on $[0, X]$ to restore $s(x)$.

CHAPTER 3. Relations of Landau-Lifschitz scattering problems and Zakharov-Shabat scattering problems

3.1 Introduction

In this chapter we study transformations $\mathfrak{F}, \mathfrak{F}_a$ from the Landau-Lifschitz scattering problem for isotropic and easy axis type of anisotropic cases to the Zakharov-Shabat scattering problem respectively. It is well known that the Landau-Lifschitz evolution equation contains the sine-Gordon, cubic Schrödinger equation as particular or limiting cases ([11]). Thus, it is natural to consider the relations of the corresponding scattering problems. Indeed, Zakharov and Takhtadzhyan formally showed the LLSP and the ZSSP are equivalent in [54]. We justify their results by defining $\mathfrak{F}$ in certain classes of coefficients Q and S, and also show that the corresponding scattering data coincide in Section 3.2.

We adapt the idea of $\mathfrak{F}$ to the ALLSP in Section 3.3. In this case, the transformation $\mathfrak{F}_a$ is not injective in general, but one can show that $\mathfrak{F}_a$ is a one to one mapping if β is fixed. Recall that β is a parameter appearing in the ALLSP. The scattering data is also invariant under this transformation.

Due to the transformations $\mathfrak{F}, \mathfrak{F}_a$ developed in Section 3.2 and 3.3, the more well known theory for the ZSSP can be adapted to the LLSP and the ALLSP. Conversely, it is worth to consider the inverse transformations, i.e. transformations from the ZSSP to the LLSP and ALLSP, since the regularity of the coefficient is decreasing under $\mathfrak{F}, \mathfrak{F}_a$. It, however, turns out that $\mathfrak{F}$ is not a surjective map. We overcome this by extending the class of coefficients of the LLSP. The forward transformation in this extended class can be understood by introducing 'step-like coefficients', which is discussed in Section 3.4. In this sense, we can define $\mathfrak{F}^{-1}$. We justify this inverse map and discuss the inverse map $\mathfrak{F}_a^{-1}$ from the ZSSP to the ALLSP as well in Section 3.5.

We mainly discuss coefficients supported on the right half line, but similar arguments presented here may give the same results for the left half line problems.

3.2 Isotropic model

Recall the LLSP

$$\frac{\partial \psi}{\partial x} = i\zeta Q\psi, \tag{3.1}$$

$$Q(x) = \begin{pmatrix} q_3(x) & q^*(x) \\ q(x) & -q_3(x) \end{pmatrix}, \quad q(x) = q_1(x) + q_2(x)i, \quad |q(x)|^2 + q_3(x)^2 = 1. \tag{3.2}$$

and the ZSSP,

$$\frac{\partial \phi}{\partial x} = i\zeta\Lambda\phi + \Lambda S\phi, \tag{3.3}$$

$$S(x) = \begin{pmatrix} 0 & s^*(x) \\ s(x) & 0 \end{pmatrix}. \tag{3.4}$$

As mentioned in Section 3.1, Zakharov and Takhtajan formally showed that the gauge equivalence of the LLSP and the ZSSP in [54]. In this section, we justify their result.

First, we define classes of coefficients Q and S of the LLSP and the ZSSP on the right half line. Let

$$\mathcal{Q}^p_{r,n} = \{Q : Q - \Lambda \in H^p_n(\mathbb{R}^+) \cap H^1(\mathbb{R}^+), Q(0) = \Lambda, |a(\zeta)| > 0 \text{ for } \zeta \in \mathbb{R}\} \tag{3.5}$$

$$\mathcal{S}^p_{r,n} = \{S : S \in L^p_n(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), |a(\zeta)| > 0 \text{ for } \zeta \in \mathbb{R}\}, \tag{3.6}$$

where, Q and S are of the forms of (3.2),(3.4) respectively, and L^p_n (H^p_n) is a weighted L^p (H^p) space, i.e. a matrix or column vector $A(x)$ is said to be in L^p_n (H^p) if $|A(x)|(1+|x|)^n \in L^p(H^p)$. Recall that for a matrix $A(x)$

$$|A(x)| = \sqrt{\lambda_{\max}(A^\dagger(x)A(x))},$$

and

$$|A(x)| = \sqrt{|A^{(1)}(x)|^2 + |A^{(2)}(x)|^2},$$

for a column vector $A(x)$. In particular, $\mathcal{S}^2_{r,0}$ is equivalent to $\mathcal{S}_r$ defined in (2.74), since $|S(x)| = |s(x)|$. Similarly, we can define $\mathcal{Q}^p_{l,n}$ and $\mathcal{S}^p_{l,n}$ on the left half line. We drop all the subscripts for notational simplicity if they are obvious. Note that the condition $Q(0) = \Lambda$ is necessary since we assume that Q has continuous extension to the whole real line and $Q - \Lambda$ is supported on the right half line. On the other hand we do not have to assume $S(0) = 0$. Similarly to $\mathcal{S}_r$, the coefficients in $\mathcal{Q}^p_{r,n}, \mathcal{S}^p_{r,n}$ can be extended by Λ, 0 on the left half line respectively.

Now we define a transformation $\mathfrak{F}$ from $\mathcal{Q}$ to $\mathcal{S}$ as following manner.

$$F_x = \frac{1}{2} Q_x Q F, \quad F(0) = I, \tag{3.7}$$

$$S = -\Lambda F^\dagger F_x. \tag{3.8}$$

Note that $QQ_x + Q_x Q = 0$ since $Q^2 = I$. Thus (3.7) can be written as

$$F_x = -\frac{1}{2} Q Q_x F, \quad F(0) = I.$$

Now we state the main theorem in this section as follows.

Theorem 3.2.1. *The map $\mathfrak{F}$ is well defined from $\mathcal{Q}$ to $\mathcal{S}$ for any $n \geq 0$ and $1 \leq p \leq \infty$. Moreover, the scattering data is invariant under the transformation $\mathfrak{F}$.*

The proof is based on Lemma 3.2.3. First we state the well known uniqueness and existence of a weak solution to first order system of equations. The proof might be found in most ODE literature for example, see [31].

Lemma 3.2.2. *For a $N \times N$ matrix $A \in L^p_n(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$,*

$$y_x = Ay, \quad y(0) = y_0,$$

has a unique weak solution ($N \times N$ matrix) in $L^\infty(\mathbb{R}^+)$. Thus y_x exists almost everywhere and $y_x \in L^p_n(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$. The weak solution is defined by the following integral equation

$$y(x) = \int_0^x A(t) y(t) dt + y_0. \tag{3.9}$$

Note that it is obvious that $\lim_{x\to\infty} y(x)$ exists. Indeed, if $x_1, x_2 \geq X$ for a sufficiently large X,

$$|y(x_1) - y(x_2)| \leq \int_{x_1}^{x_2} |A(t)| ||y||_\infty < \varepsilon.$$

Lemma 3.2.3. *For a coefficient $Q \in \mathcal{Q}$ in the LLSP, the following equations are equivalent.*

(1)

$$F_x = -\frac{1}{2} Q Q_x F = \frac{1}{2} Q_x Q F, \quad F(0) = I. \tag{3.10}$$

(2) F_x exists almost everywhere,

$$\begin{aligned} QF &= F\Lambda, & (3.11)\\ F^\dagger F &= I, & (3.12)\\ \mathcal{D}(F^\dagger F_x) &= 0, & (3.13)\\ F(0) &= I. & (3.14) \end{aligned}$$

(3) F_x exists almost everywhere,

$$\begin{aligned} QF &= F\Lambda, & (3.15)\\ QF_x &= -F_x\Lambda, & (3.16)\\ F(0) &= I. & (3.17) \end{aligned}$$

Here, $\mathcal{D}(A) = diag(A_{11}, A_{22})$ for a 2×2 matrix A.

Proof.

(1)→(2); From the equation (3.10), it is easy to check that $(F^\dagger F)_x = 0$. Together with the initial condition $F(0) = I$, we have (3.12). Multiplying Q and Λ to the left and the right of each side of (3.10) respectively yields

$$\begin{aligned} QF_x\Lambda &= -\frac{1}{2} Q_x F\Lambda, \quad (QF\Lambda)(0) = I,\\ QF_x\Lambda &= \frac{1}{2} Q Q_x Q F \Lambda. \end{aligned}$$

Since $(QF\Lambda)_x = Q_x F\Lambda + QF_x\Lambda$,

$$(QF\Lambda)_x = -QF_x\Lambda = -\frac{1}{2} Q Q_x (QF\Lambda), \quad (QF\Lambda)(0) = I.$$

Thus $QF\Lambda$ solves (3.10). The uniqueness of ODE system gives

$$QF\Lambda = F, \quad \text{or } QF = F\Lambda.$$

Finally,

$$\begin{aligned}
F^\dagger F_x &= -\frac{1}{2}F^\dagger QQ_xF \\
&= -\frac{1}{2}\Lambda F^\dagger Q_xF \\
&= -\frac{1}{2}\Lambda F^\dagger(F_x\Lambda - QF_x) \\
&= -\frac{1}{2}\Lambda F^\dagger F_x\Lambda + \frac{1}{2}F^\dagger F_x. \\
F^\dagger F_x &= -\Lambda F^\dagger F_x\Lambda.
\end{aligned}$$

Since $\mathcal{D}(A) = \mathcal{D}(\Lambda A\Lambda)$ for any matrix A, $\mathcal{D}(F^\dagger F_x) = 0$.

(2)→(3); (3.13) implies that $F^\dagger F_x$ is an off-diagonal matrix. Then

$$F^\dagger F_x\Lambda = -\Lambda F^\dagger F_x.$$

Since $Q^2 = I$,

$$\begin{aligned}
F^\dagger QQF_x\Lambda &= -\Lambda F^\dagger F_x \\
QF_x &= -QF\Lambda F^\dagger F_x\Lambda \\
QF_x &= -F_x\Lambda.
\end{aligned}$$

(3)→(1); From (3.15), (3.16)

$$\begin{aligned}
Q_xF + QF_x &= F_x\Lambda = -QF_x \\
Q_xF &= -2QF_x \\
F_x &= -\frac{1}{2}QQ_xF = \frac{1}{2}Q_xQF.
\end{aligned}$$

□

Now we ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. It is obvious that $Q_x Q \in L^1(\mathbb{R}^+) \cap L^p_n(\mathbb{R}^+)$ for $Q \in \mathcal{Q}$ since Q is bounded. Lemma 3.2.2 implies that F exists uniquely and $F_x, -\Lambda F^\dagger F_x \in L^1(\mathbb{R}^+) \cap L^p_n(\mathbb{R}^+)$. Since $-\Lambda F^\dagger F_x = (-\Lambda F^\dagger F_x)^\dagger$ from (3.12), (3.13), S defined by (3.8) is in the class $\mathcal{S}$.

Next, we show that the scattering data for Q in the LLSP coincides with the scattering data for $\mathfrak{F}Q$ in the ZSSP. Suppose that $F \to F_\infty$ as $x \to \infty$. Since $Q \to \Lambda$ as $x \to \infty$, (3.11) implies $\Lambda F_\infty = F_\infty \Lambda$, thus F_∞ should be a diagonal matrix. Moreover, Liouville's formula gives

$$\det F(x) = \det F(0) \exp(\int_0^x \mathrm{trace}(\frac{1}{2} Q_x Q)) = 1,$$

since $\mathrm{trace}(Q_x Q) = 0$. Hence, F_∞ should be a diagonal matrix such that

$$F_\infty = \begin{pmatrix} f_\infty & 0 \\ 0 & f_\infty^* \end{pmatrix}, \quad |f_\infty| = 1. \tag{3.18}$$

Recall the definition of the transition matrix $\mathcal{T}$ from the Jost solutions $J^\pm$ to the LLSP.

$$J^+ = J^- \mathcal{T}. \tag{3.19}$$

Multiplying $F^\dagger$ to the left and F_∞ to the right of (3.19) give

$$F^\dagger J^+ F_\infty = F^\dagger J^- \mathcal{T} F_\infty.$$

It is not hard to show that $F^\dagger J^+ F_\infty$ and $F^\dagger J^-$ solve (3.3) for S given by (3.8). Indeed, for a solution ψ to (3.1)

$$\begin{aligned} (F^\dagger \psi)_x &= F_x^\dagger \psi + F^\dagger \psi_x, \\ &= F_x^\dagger F (F^\dagger \psi) + F^\dagger \phi_x \quad \text{by (3.12)}, \\ &= F_x^\dagger F (F^\dagger \psi) + i\zeta\Lambda(F^\dagger \phi) \quad \text{by (3.11)}. \end{aligned}$$

Since F_∞ is a diagonal matrix, $F^\dagger J^+ F_\infty$ and $F^\dagger J^-$ are solutions of (3.3). Furthermore, $F^\dagger J^+ F_\infty$ and $F^\dagger J^-$ have the following asymptotic behaviors,

$$\begin{aligned} F^\dagger J^+ F_\infty &\to e^{i\zeta\Lambda x} \quad \text{as } x \to \infty, \\ F^\dagger J^- &= e^{i\zeta\Lambda x} \quad \text{for } x < 0, \end{aligned}$$

by extending F to I for $x<0$. By the uniqueness of Jost solutions in the ZSSP, $F^\dagger J^+ F_\infty$ and $F^\dagger J^-$ are the right and left Jost solutions for $\mathfrak{F}Q$ in the ZSSP respectively. Thus, $\mathcal{T}F_\infty$ is the transition matrix for the ZSSP. The structure of F_∞, (3.18), guaranties the preservation of the scattering data under $\mathfrak{F}$. $\square$

We remark that $F_\infty = I$. It is not hard to show that $a(\zeta)\to 1$ as $\zeta\to\infty$ in the upper half plane ([25]). Then it follows from the spectral representation, (1.21) that the transition matrices for the LLSP and the ZSSP coincide. That is, $\mathcal{T}=\mathcal{T}F_\infty$, thus $F_\infty = I$.

As mentioned earlier, one of the benefits of the transformation $\mathfrak{F}$ is the adaptation of the more well known theory of the ZSSP to the LLSP. For this end, we have to restore Q from $\mathfrak{F}Q$.

Theorem 3.2.4. *The transformation $\mathfrak{F}$ is injective, and $\mathfrak{F}^{-1} : \mathcal{S}\supset\mathfrak{F}(\mathcal{Q})\to\mathcal{Q}$ is given by*

$$F_x^\dagger = \Lambda S F^\dagger, \quad F(0)=I, \tag{3.20}$$

$$Q = F\Lambda F^\dagger. \tag{3.21}$$

Proof. Suppose that $\mathfrak{F}Q=\mathfrak{F}P$ for $Q,P\in\mathcal{Q}$. Then, there are F and G such that $F(0)=G(0)=I$, $\mathfrak{F}Q=-\Lambda F^\dagger F_x$, $\mathfrak{F}P=-\Lambda G^\dagger G_x$ and $F^\dagger F_x = G^\dagger G_x$. So we have

$$F_x^\dagger = G_x^\dagger G F^\dagger, \quad F(0)=I.$$

Consider the following integral representation,

$$F^\dagger(x) = I + \int_0^x G_x^\dagger(y)G(y)F^\dagger(y)dy.$$

Integration by part gives

$$\int_0^x G^\dagger(y)(GF^\dagger)_x(y)dy = 0.$$

This holds for almost all x, thus $G^\dagger(GF^\dagger)_x=0$, or $GF^\dagger = C$ for some constant matrix C. The initial condition $G(0)=I$ implies $G=F$, thus $Q=P$.

One can easily check (3.20) and (3.21) from Lemma 3.2.3 as long as $S\in\mathfrak{F}(\mathcal{Q})$. $\square$

Figure 3.1 shows a reconstructed coefficient $Q(x)$ in the LLSP via the transformation $\mathfrak{F}$. Similarly to the examples of the ZSSP given in Section 2.6, the left reflection coefficient $L(\zeta)$ is directly computed via an ODE solver. From given $L(\zeta)$, we extract a non pure imaginary eigenvalue $\zeta_1 = 12.818554682013753 + 0.507439920154443i$, and reconstruct $s(x)$ by the Darboux transformation. Finally, $Q(x)$ is restored from (3.20) and (3.21).

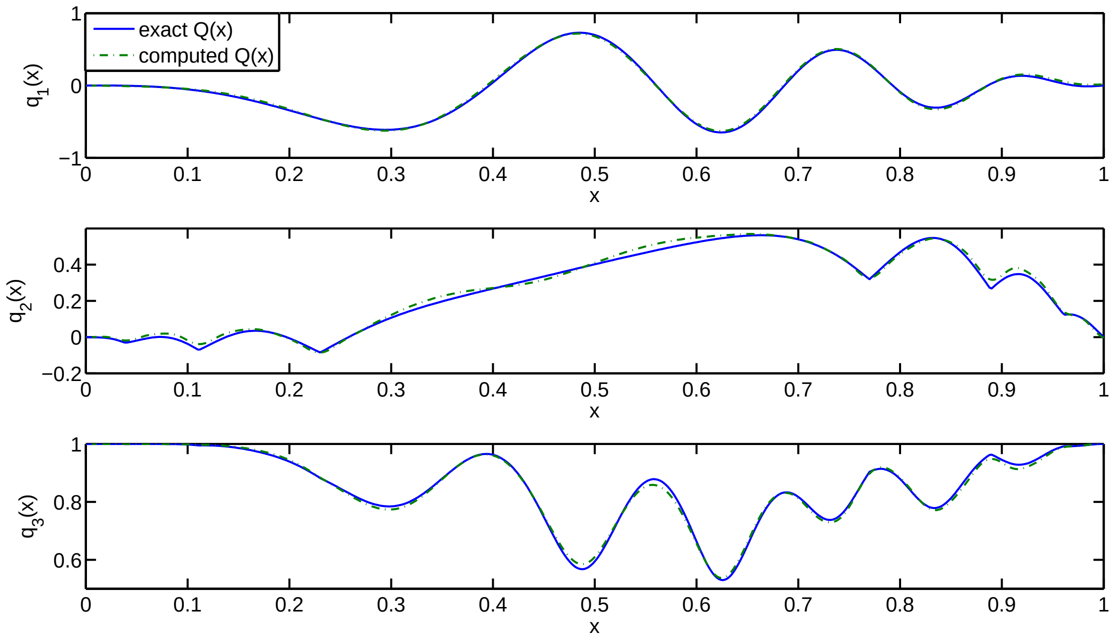

Figure 3.1 Reconstructed Q via the transformation $\mathfrak{F}$

3.3 Anisotropic model of easy axis

Consider the ALLSP with parameter[1] $\beta \in \mathbb{R}$,

$$\frac{\partial \psi}{\partial x} = i\zeta Q\psi - \beta\Lambda\mathcal{D}^c Q\psi,$$

where, $\mathcal{D}^c Q = Q - \mathcal{D}Q$, i.e.

$$\mathcal{D}^c Q = \begin{pmatrix} 0 & q^* \\ q & 0 \end{pmatrix}.$$

Similarly to the LLSP, we can develop a transformation $\mathfrak{F}_a$ from $\mathcal{Q} \times \mathbb{R}$ to $\mathcal{S}$.

[1]Recall that the LLSP is the ALLSP with $\beta = 0$

Define ϕ such that $\psi = F\phi$ for a solution ψ to the LLSP, (3.1) and F to (3.10) for a fixed $Q \in \mathcal{Q}$. Then, by Lemma 3.2.3

$$
\begin{aligned}
F_x\phi + F\phi_x &= i\zeta QF\phi - \beta\Lambda\mathcal{D}^c QF\phi, && (3.22)\\
F^\dagger F_x\phi + \phi_x &= i\zeta F^\dagger QF\phi - \beta F^\dagger\Lambda\mathcal{D}^c QF\phi, && (3.23)\\
\phi_x &= i\zeta\Lambda\phi + \Lambda S\phi - \beta F^\dagger\Lambda\mathcal{D}^c QF\phi. && (3.24)
\end{aligned}
$$

The following lemma shows that the equation (3.24) is the ZSSP with a coefficient in $\mathcal{S}$.

Lemma 3.3.1. *Let $\tilde{S} := S - \beta\Lambda F^\dagger\Lambda\mathcal{D}^c QF$. Then $\tilde{S} \in \mathcal{S}$ provided $(Q,\beta) \in \mathcal{Q}\times\mathbb{R}$.*

Proof. Obviously $\tilde{S} \in L^p_n \cap L^1$ is supported on the right half line. We need to show that $\tilde{S}$ is of the form of (3.4). For this aim, we first claim that $\tilde{S}$ is a Hermitian matrix. Indeed,

$$
\begin{aligned}
(\Lambda\mathcal{D}^c Q)^\dagger &= \mathcal{D}^c Q\Lambda = -\Lambda\mathcal{D}^c Q,\\
(F^\dagger\Lambda\mathcal{D}^c QF)^\dagger &= -F^\dagger\Lambda QF.
\end{aligned}
$$

Thus $\Lambda\tilde{S}$ is a skew-Hermitian and $\tilde{S}$ is a Hermitian matrix as desired.

Next we show that $F^\dagger\Lambda\mathcal{D}^c QF$ is an off-diagonal matrix.

$$
\begin{aligned}
\Lambda &= F^\dagger QF = F^\dagger(q_3\Lambda + \mathcal{D}^c Q)F,\\
&= q_3F^\dagger\Lambda F + F^\dagger\mathcal{D}^c QF,\\
&= q_3F^\dagger\Lambda F + (F^\dagger\Lambda F)(F^\dagger\Lambda\mathcal{D}^c QF),\\
F^\dagger\Lambda\mathcal{D}^c QF &= -q_3I + (F^\dagger\Lambda F)\Lambda.
\end{aligned}
$$

Since $\mathcal{D}(Q\Lambda) = q_3I$, if $\mathcal{D}(F^\dagger\Lambda F) = \mathcal{D}(F\Lambda F^\dagger)$, then we prove the lemma. Indeed, for $i = 1,2$

$$
\begin{aligned}
\sum_{n,m} F^\dagger_{in}\Lambda_{nm}F_{mi} - \sum_{n',m'} F_{in'}\Lambda_{n'm'}F^\dagger_{m'i} &= (F^*_{1i}F_{1i} + F_{i2}F^*_{i2}) - (F^*_{2i}F_{2i} + F_{i1}F^*_{i1}),\\
&= 1 - 1 = 0.
\end{aligned}
$$

□

The scattering data is preserved under this transformation as well. The proof is exactly same as the case of the isotropic model. Now we can state the following theorem to describe the transformation $\mathfrak{F}_a$ from (Q,β) in the ALLSP to $\tilde{S}$ in the ZSSP.

Theorem 3.3.2. *The map*

$$\begin{aligned}\mathfrak{F}_a : \quad \mathcal{Q} \times \mathbb{R} &\to \mathcal{S} \\ (Q, \beta) &\mapsto \tilde{S}\end{aligned}$$

is well defined by

$$F_x = \frac{1}{2} Q_x Q F, \quad F(0) = I, \tag{3.25}$$

$$S = -\Lambda F^{\dagger} F_x, \tag{3.26}$$

$$\tilde{S} = S - \beta \Lambda F^{\dagger} \Lambda \mathcal{D}^c Q F. \tag{3.27}$$

The scattering data for the ALLSP with (Q, β) coincides with one for the ZSSP with $\tilde{S}$.

Generally $\mathfrak{F}_a$ is not injective, that is, for some $(Q, \beta_1) \neq (P, \beta_2)$ may be transformed to the same $\tilde{S}$. See Section 3.5 for details. However, one can show that $\mathfrak{F}_a|_{\beta=\beta_0}$ is a one to one map. In other word, $\mathfrak{F}_a(Q, \beta_0) = \mathfrak{F}_a(P, \beta_0)$ implies $Q = P$ in $\mathcal{Q}$ for any fixed $\beta_0 \in \mathbb{R}$.
Rewrite (3.25)-(3.27) by aid of Lemma 3.2.3,

$$F_x = -F \Lambda \tilde{S} - \beta_0 \Lambda \mathcal{D}^c (F \Lambda F^{\dagger}) F, \quad F(0) = I. \tag{3.28}$$

To show the uniqueness, it is enough to show (3.28) has a unique solution. Let $\mathcal{H}(F) = -F \Lambda \tilde{S} - \beta_0 \Lambda \mathcal{D}^c (F \Lambda F^{\dagger}) F$. Then,

$$|\mathcal{H}(F) - \mathcal{H}(G)| \leq (|\tilde{S}| + 2\beta_0) |F - G|.$$

Thus (3.28) has at most one solution.

We remark that for a solution $F(x)$ to (3.28) $\lim_{x \to \infty} F(x)$ exists as long as $\tilde{S} \in L^1(\mathbb{R}^+)$. The proof is very similar to one for the Darboux transformation discussed in Section 2.4. It is not difficult to show that any solution to

$$F_x = -\beta_0 \Lambda \mathcal{D}^c (F \Lambda F^{\dagger}) F$$

is a constant or converges to a constant matrix F_∞ exponentially. Then it follows from $|-F \Lambda \tilde{S}| \leq |\tilde{S}||F|$ that the solution F to (3.28) is a constant matrix on an interval (X, ∞) or converges to F_∞ exponentially.

Theorem 3.3.3. *$\mathfrak{F}_a(\cdot, \beta_0)$ is an injective map from $\mathcal{Q}$ onto $\mathfrak{F}_a(\mathcal{Q}, \beta_0) \subset \mathcal{S}$ for a fixed $\beta_0 \in \mathbb{R}$. For $\tilde{S} \in \mathfrak{F}_a(\mathcal{Q}, \beta_0)$, $\mathfrak{F}_a^{-1}(\tilde{S})$ is given by (3.28) and (3.21).*

Note that $\mathfrak{F}_a(\mathcal{Q}, \beta_0) \subsetneq \mathcal{S}$ in general. More details can be found in Section 3.5.

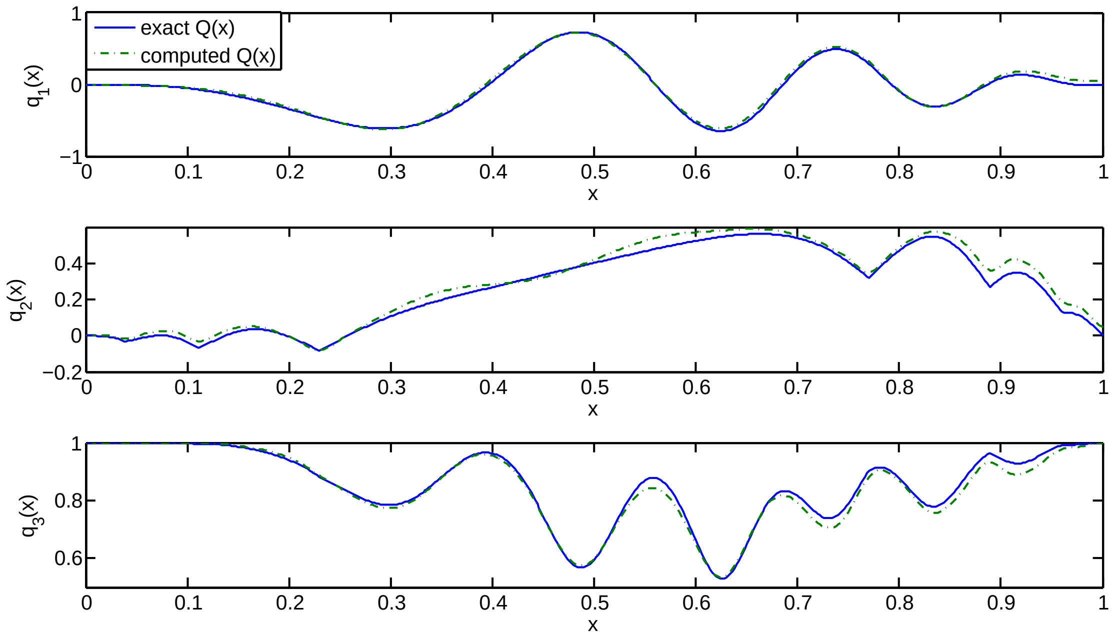

Figure 3.2 Reconstructed Q via the transformation $\mathfrak{F}_a$ for $\beta = 1$

Figure 3.2 shows a reconstructed coefficient $Q(x)$ of the ALLSP with $\beta = 1$ from $L(\zeta)$ via $\mathfrak{F}_a$. First, we extract the eigenvalue $\zeta_1 = 12.805318352723857 + 0.512585631071842i$ from $\widehat{L}(t)$, and reconstruct $s^{[0]}(x)$ from $L^{[0]}(\zeta)$. Then we use (3.28) and (3.21) to restore $Q(x)$ together with $s(x)$ which is computed form $s^{[0]}(x)$ and the Darboux transformation.

3.4 Step-like coefficients

A coefficient $\Omega(x, \zeta)$ of (1.1) is said to be a step-like coefficient if two limits at $\pm\infty$ are different, that is

$$\lim_{x \to \infty} \Omega(x, \zeta) \neq \lim_{x \to -\infty} \Omega(x, \zeta).$$

The step-like coefficient theory for the Schrödinger equation is intensively studied, for instance, see [19, 47] and references therein. In this section, we discuss the direct problem for the LLSP with a step-like coefficient briefly and define a transformation of the LLSP with a step-like coefficient to the ZSSP.

As discussed in Section 1.1, the Jost solutions and the transition matrix are well defined and they have the same properties as the case of non step-like coefficients . That is, for a coefficient Q in the LLSP on the right half line satisfying

$$\lim_{x\to\infty} Q(x) = Q_\infty, \qquad Q(x) = Q_0 \quad x < 0,$$

the Jost solutions are defined by

$$\begin{aligned} J^+(x,\zeta) &\rightarrow e^{i\zeta Q_\infty x}, \quad x \to \infty, \\ J^-(x,\zeta) &= e^{i\zeta Q_0 x}, \quad x < 0. \end{aligned}$$

Then the transition matrix,

$$\mathcal{T} = \begin{pmatrix} a(\zeta) & -b(\zeta^*)^* \\ b(\zeta) & a(\zeta^*)^* \end{pmatrix}$$

exists such that

$$J^+(x,\zeta) = J^-(x,\zeta)\mathcal{T}(\zeta).$$

Moreover, $a(\zeta)$ and $b(\zeta)$ are analytic in $\mathbb{C}^+$ and $\mathbb{C}^-$ respectively, if $Q(x) \to Q_\infty$ sufficiently fast as $x \to \infty$. The reflection and transmission coefficients are defined by (1.14),

$$L(\zeta) = \frac{b(\zeta)}{a(\zeta)}, \quad R(\zeta) = \frac{b^*(\zeta)}{a(\zeta)}, \quad T(\zeta) = \frac{1}{a(\zeta)}, \quad \zeta \in \mathbb{R}.$$

The properties of bound states, however, are different as the case of non step-like coefficients. Suppose that a unitary matrix F_∞ satisfies

$$Q_\infty F_\infty = F_\infty \Lambda.$$

Then, from the LLSP, (3.1), we have

$$\begin{aligned} F_\infty^\dagger \frac{\partial \psi}{\partial x} &= i\zeta F_\infty^\dagger Q \psi, \\ \frac{\partial (F_\infty^\dagger \psi)}{\partial x} &= i\zeta F_\infty^\dagger Q F_\infty (F_\infty^\dagger \psi). \end{aligned}$$

Thus, $F_\infty^\dagger \psi$ solves the LLSP with a coefficient $F_\infty^\dagger Q F_\infty$. From the definition of F_∞,

$$\lim_{x\to\infty} F_\infty^\dagger Q F_\infty = \Lambda.$$

Hence, without loss of generality, we assume $Q_\infty = \Lambda$ in the LLSP with a step-like coefficient.

Similarly to F_∞, we define a unitary matrix F_0 corresponding to Q_0 as

$$Q_0 F_0 = F_0 \Lambda, \quad F_0 = \begin{pmatrix} f_0 & -g_0^* \\ g_0 & f_0^* \end{pmatrix}. \tag{3.29}$$

For $x < 0$, the left Jost solution $J^-(x,\zeta)$ is given by

$$\begin{aligned} J^-(x,\zeta) &= F_0 e^{i\zeta\Lambda x} F_0^\dagger & (3.30)\\ &= \begin{pmatrix} |f_0|^2 e^{i\zeta x} + |g_0|^2 e^{-i\zeta x} & f_0 g_0^* e^{i\zeta x} - f_0 g_0^* e^{-i\zeta x} \\ f_0^* g_0 e^{i\zeta x} - f_0^* g_0 e^{-i\zeta x} & |f_0|^2 e^{-i\zeta x} + |g_0|^2 e^{i\zeta x} \end{pmatrix}. & (3.31) \end{aligned}$$

Substitution (3.31) into (1.12) gives

$$\mu(x,\zeta) = \begin{pmatrix} (f_0^* a(\zeta) + g_0^* b(\zeta)) f_0 e^{i\zeta x} + (g_0 a(\zeta) - f_0 b(\zeta)) g_0^* e^{-i\zeta x} \\ (f_0^* a(\zeta) + g_0^* b(\zeta)) g_0 e^{i\zeta x} - (g_0 a(\zeta) - f_0 b(\zeta)) f_0^* e^{-i\zeta x} \end{pmatrix}, \ x < 0. \tag{3.32}$$

Suppose that the LLSP has a bound state at $\zeta = \zeta_0 \in \mathbb{C}^+$. Then, the bound state should be a scalar multiple of $\mu(x,\zeta_0)$ because we assume that $Q_\infty = \Lambda$. Since $\zeta_0 \in \mathbb{C}^+$, from (3.32)

$$f_0^* a(\zeta_0) + g_0^* b(\zeta_0) = 0, \quad \zeta_0 \in \mathbb{C}^+. \tag{3.33}$$

Similarly to the argument in Section 1.1, one can show that (3.33) is the necessary and sufficient condition for the bound state.

We define a normalizing constant $C_{r,0}$ as

$$C_{r,0} = \frac{-g_0 a(\zeta_0) + f_0 b(\zeta_0)}{f_0^* \dot{a}(\zeta_0) + g_0^* \dot{b}(\zeta_0)}, \tag{3.34}$$

provided the order of zero ζ_0 of (3.33) is one. Here, $\dot{a}(\zeta) = \dfrac{da(\zeta)}{d\zeta}$.

We remark that the bound state data $\{\zeta_n, C_{r,n}\}$ is uniquely determined from (3.33) and (3.34) even though F_0 is not unique. Let G_0 be another unitary matrix such that

$$Q_0 G_0 = G_0 \Lambda.$$

Then,

$$\begin{aligned} F_0^\dagger G_0 \Lambda &= F_0^\dagger Q_0 G_0 \\ &= (F_0 \Lambda)^\dagger G_0 \\ &= \Lambda F_0^\dagger G_0, \end{aligned}$$

that is, $F_0^\dagger G_0$ should commute with Λ. This implies that for some $|c| = 1$,

$$G_0 = F_0 C, \quad C = \begin{pmatrix} c & 0 \\ 0 & c^* \end{pmatrix}, \tag{3.35}$$

because any matrix which commute with Λ should be a diagonal matrix and $F_0^\dagger G_0$ is a unitary matrix. From (3.35), it is easy to check that the bound state data given by (3.33) and (3.34) is not changed. Thus, they may be represented in terms of Q_0. Indeed, by a specific choice of F_0, for example,

$$F_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+p_0} & -\dfrac{q_0^*}{\sqrt{1+p_0}} \\ \dfrac{q_0}{\sqrt{1+p_0}} & \sqrt{1+p_0} \end{pmatrix}, \quad q_0 = q_1(0) + q_2(0)i, \;\; p_0 = q_3(0), \tag{3.36}$$

for $p_0 \neq -1$. Then one can easily check that the eigenvalue $\zeta_0 \in \mathbb{C}^+$ is a zero of

$$(1+p_0)a(\zeta) + q_0^* b(\zeta) = 0,$$

and

$$C_{r,0} = \frac{-q_0 a(\zeta_0) + (1+p_0) b(\zeta_0)}{(1+p_0)\dot{a}(\zeta_0) + q_0^* \dot{b}(\zeta_0)}.$$

In the case of $p_0 = -1$ or $Q_0 = -\Lambda$, one can define F_0 as a limit of (3.36), i.e. $F_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now we are ready to construct a transformation $\mathfrak{F}_s$ from step-like coefficients in the LLSP to coefficients in the ZSSP. Define $\mathcal{Q}_{r,n}^{s,p}$ by

$$\mathcal{Q}_{r,n}^{s,p} = \{Q : Q - \Lambda \in H_n^p(\mathbb{R}^+) \cap H^1(\mathbb{R}^+), \;\; f_0^* a(\zeta) + g_0^* b(\zeta) \neq 0, \;\; \text{for } \zeta \in \mathbb{R}\}, \tag{3.37}$$

where $Q(x)$ is defined by $Q(0)$ for $x < 0$. The condition $f_0^* a(\zeta) + g_0^* b(\zeta) \neq 0$ is a generic condition for step-like coefficients. It may be rewritten as

$$\begin{aligned} &(1+p_0)a(\zeta) + q_0^* b(\zeta) \neq 0, \quad \text{for } \zeta \in \mathbb{R} \qquad && \text{if } p_0 \neq -1, \\ &q_0 a(\zeta) + (1-p_0)b(\zeta) \neq 0, \quad \text{for } \zeta \in \mathbb{R} \qquad && \text{if } p_0 \neq 1. \end{aligned}$$

Trivially $\mathcal{Q}^p_{r,n}$ defined in (3.5) is a strict subset of $\mathcal{Q}^{s,p}_{r,n}$. The map $\mathfrak{F}_s$ from $\mathcal{Q}^{s,p}_{r,n}$ to $\mathcal{S}^p_{r,n}$, the class of coefficients of the ZSSP defined in (3.6), is defined by a similar manner to $\mathfrak{F}$. The only different point is the initial condition of (3.7). That is, $\mathfrak{F}_s Q$ is given by

$$F_x = \frac{1}{2} Q_x Q F, \quad F(0) = F_0, \tag{3.38}$$

$$S = -\Lambda F^\dagger F_x. \tag{3.39}$$

Here, $F(0)$ should be F_0 defined in (3.29) instead of I.

The scattering data is not preserved under this operation. Let $\lim_{x\to\infty} F(x) = F_\infty$. We showed F_∞ is a diagonal matrix in (3.18). Thus, for the Jost solutions $J^\pm$ and the transition matrix $\mathcal{T}$,

$$F^\dagger(x) J^+(x,\zeta) F_\infty = F^\dagger(x) J^-(x,\zeta) \mathcal{T}(\zeta) F_\infty,$$

or

$$F^\dagger(x) J^+(x,\zeta) F_\infty = F^\dagger(x) J^-(x,\zeta) F_0 F_0^\dagger \mathcal{T}(\zeta) F_\infty.$$

The properties of F insure that

$$\begin{aligned} F^\dagger(x) J^+(x,\zeta) F_\infty &\to e^{i\zeta\Lambda x}, \quad x \to \infty, \\ F^\dagger(x) J^-(x,\zeta) F_0 &= e^{i\zeta\Lambda x}, \quad x < 0. \end{aligned}$$

Thus,

$$F_0^\dagger \mathcal{T}(\zeta) F_\infty = \begin{pmatrix} f_0^* f_\infty a(\zeta) + g_0^* f_\infty b(\zeta) & g_0^* f_\infty^* a^*(\zeta^*) - f_0^* f_\infty^* b^*(\zeta^*) \\ -g_0 f_\infty a(\zeta) + f_0 f_\infty b(\zeta) & f_0 f_\infty^* a^*(\zeta^*) + g_0 f_\infty^* b^*(\zeta^*) \end{pmatrix} \tag{3.40}$$

is the transition matrix for the ZSSP with $\mathfrak{F}_s Q$. In particular, the transformed left scattering data $\{\breve{L}(\zeta), \breve{\zeta}_n, \breve{C}_{r,n}\}_{n=1}^N$ is given by

$$\begin{aligned}
\breve{L}(\zeta) &= \frac{-g_0 a(\zeta) + f_0 b(\zeta)}{f_0^* a(\zeta) + g_0^* b(\zeta)} \\
&= \frac{-g_0 + f_0 L(\zeta)}{f_0^* + g_0^* L(\zeta)}, \qquad (3.41) \\
\breve{\zeta}_n &= \zeta_n, \qquad (3.42) \\
\breve{C}_{r,n} &= C_{r,n}. \qquad (3.43)
\end{aligned}$$

Note that $\mathfrak{F}_s Q$ is unique up to constant multiplication. Indeed, it depends on F_0, the initial condition of (3.38). We showed the unitary matrices F_0 and G_0 satisfying (3.29) are related by (3.35) for some $|c| = 1$. The solutions $F(x)$ and $G(x)$ to (3.38) with initial condition F_0 and G_0 are also related by $G(x) = F(x)C$, and $\mathfrak{F}_s Q$ given by

$$S_{G_0} = -\Lambda G^\dagger G_x, \quad S_{F_0} = -\Lambda F^\dagger F_x,$$

respectively. Thus

$$\begin{aligned}
S_{G_0} &= -\Lambda C^\dagger F^\dagger F_x C \qquad (3.44) \\
&= C^\dagger S_{F_0} C \qquad (3.45) \\
s_{G_0} &= c^2 s_{F_0}. \qquad (3.46)
\end{aligned}$$

Here, s_{G_0}, s_{F_0} are the $(1,2)$ entries of S_{G_0}, S_{F_0} respectively. This non-uniqueness can be fixed by a specific choice of F_0, e.g. (3.36).

We summarize above argument as the following theorem.

Theorem 3.4.1. *The transformation $\mathfrak{F}_s$ from $\mathcal{Q}_{r,n}^{s,p}$ to $\mathcal{S}_{r,n}^p$ is well defined by (3.38) and (3.39), where F_0 is a unitary matrix such that $Q_0 F_0 = F_0 \Lambda$. The left reflection coefficient and the bound state data under this transformation are given by (3.41)-(3.43). $\mathfrak{F}_s Q$ depends on the initial condition F_0 up to constant multiplication. For a specific choice of F_0, for example, (3.36) $\mathfrak{F}_s$ is injective and $\mathfrak{F}_s^{-1} S$ is given by (3.20) with $F(0) = F_0$ and (3.21).*

Figure 3.3 shows a reconstructed $Q(x)$ with the initial condition F_0 given by (3.36). From the given data $L(\zeta)$ and Q_0, we restore $s(x)$ from $\breve{L}(\zeta)$ which is defined in (3.41). Again the

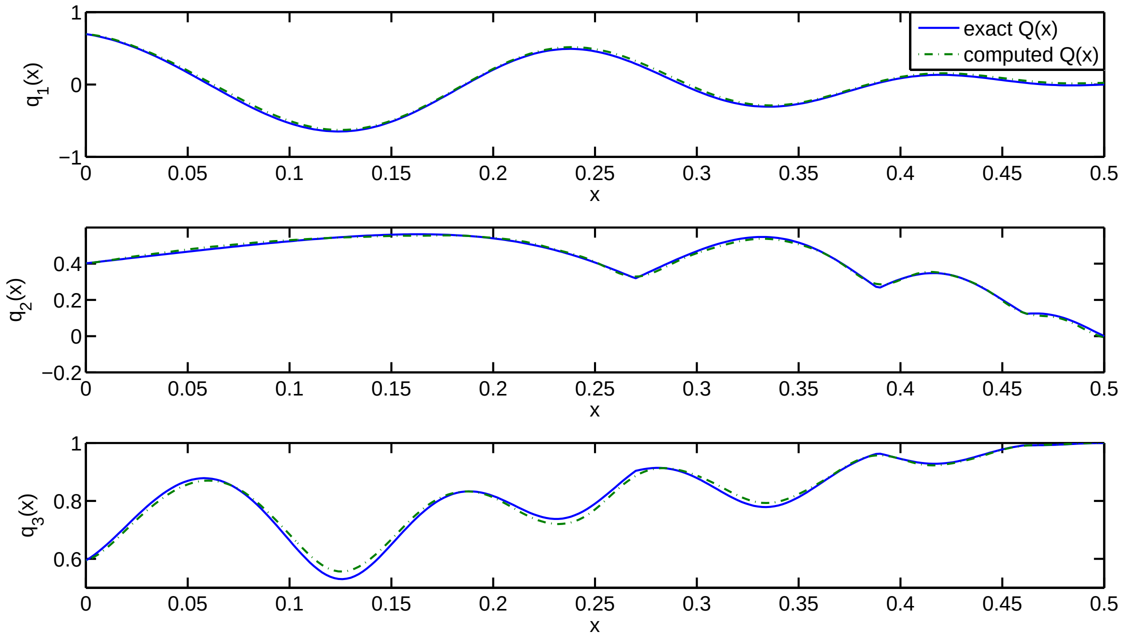

Figure 3.3 Reconstructed Q via the transformation $\mathfrak{F}_s$ with F_0

layer stripping method and the Darboux transformation are used. $Q(x)$ is computed from $s(x)$ by (3.20) with $F(0) = F_0$ and (3.21). The coefficients s_{F_0} and s_{G_0} for $G_0 = F_0 C$ with $c = e^{\pi i/4}$ are showed in Figure 3.4. As we discussed in (3.46), $s_{G_0} = i s_{F_0}$.

3.5 Inverse map; map from S to Q

In Section 3.2, we defined a map $\mathfrak{F}$ from the isotropic model of LLSP to the ZSSP and showed it is a one to one map. However, it is not an onto map from $\mathcal{Q}$ to $\mathcal{S}$[2]. Suppose $S \in \mathcal{S}$ is given. Then the corresponding F and Q can be computed from (3.20) and (3.21), but in general, the computed Q is not in $\mathcal{Q}$. For simplicity of current discussion, assume that S is supported on $[0, X]$. Since $S = 0$ on (X, ∞), $F^\dagger$ is a constant matrix on this interval, say $F_\infty^\dagger$. Then $Q = F_\infty \Lambda F_\infty^\dagger$ need not be Λ on (X, ∞). Indeed, this is possible only if F_∞ is a diagonal

[2]We omit the subscripts r, n, p from $\mathcal{Q}_{r,n}^p$ and $\mathcal{S}_{r,n}^p$

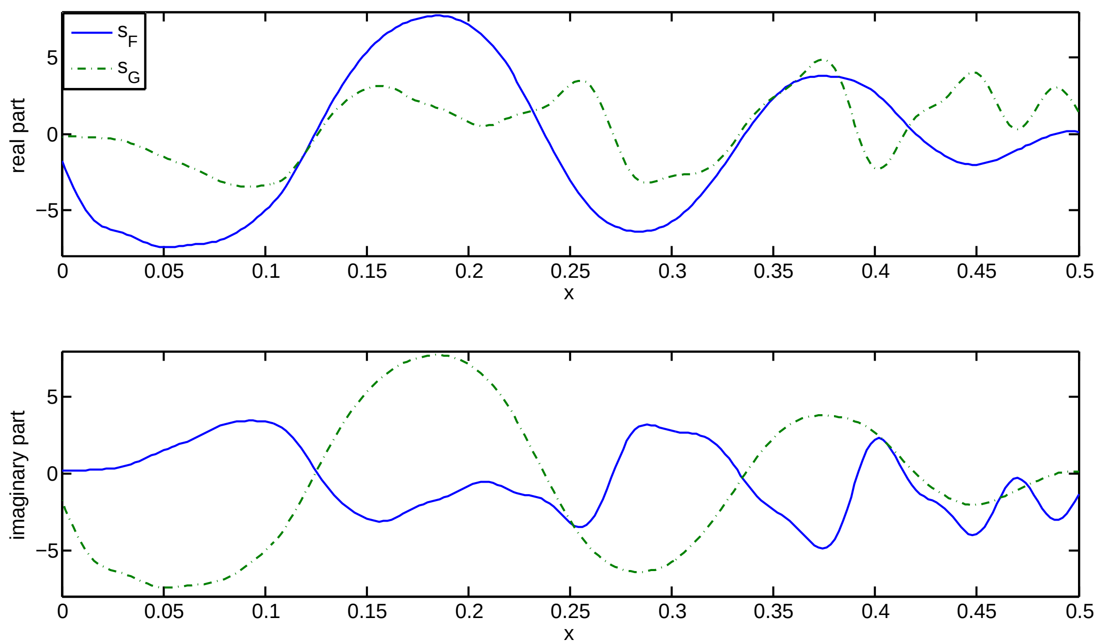

Figure 3.4 Coefficients s in the transformed ZSSP

matrix. For instance, for given $s(x) = \chi_{[0,1]}(x)$,

$$F^{\dagger}(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

solves (3.20). Thus for $x > 1$,

$$Q = F_{\infty} \Lambda F_{\infty}^{\dagger} = \begin{pmatrix} \cos 2 & \sin 2 \\ \sin 2 & -\cos 2 \end{pmatrix} \neq \Lambda.$$

This example shows that $\mathfrak{F}(\mathcal{Q})$ is a strict subset of $\mathcal{S}$.

The range of $\mathfrak{F}$ is characterized by $L(0)$ as Zakharov and Takhtadzhyan pointed out in [54]. In the LLSP, $J^{+}(x,0) = J^{-}(x,0) = I$, and thus $L(0) = 0$. Conversely, we claim that $L(0) = 0$ in the ZSSP with $S \in \mathcal{S}$ is a sufficient condition for $S \in \mathfrak{F}(\mathcal{Q})$.

Consider the ZSSP at $\zeta = 0$,

$$\phi_x = \Lambda S \phi. \tag{3.47}$$

The left Jost solution, $J^-(x,0)$ solves (3.47) with initial condition $\phi(0,0)=I$. Since $J^+(x,0)\to I$ as $x\to\infty$, from $J^+(x,0)=J^-(x,0)\mathcal{T}(0)$

$$I=\lim_{x\to\infty}J^-(x,0)\mathcal{T}(0). \tag{3.48}$$

If $L(0)=0$, then $\mathcal{T}(0)=\begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}$ for some $|a|=1$. Since (3.47) with the initial condition $\phi(0,0)=I$ and (3.20) are equivalent, $F^\dagger=J^-(x,0)$ or $F_\infty=\mathcal{T}(0)$. We showed $\mathcal{T}(0)$ is a diagonal matrix and this is a sufficient condition for $Q=F\Lambda F^*\to\Lambda$. Moreover, the scattering data is invariant under this transformation.

However, $L(0)$ is allowed to be any complex number in the ZSSP. Consider a piecewise constant coefficient $s(x)=s\chi_{[0,1]}(x)$ for $s=re^{i\theta}\in\mathbb{C}$. It is easy to check

$$L(\zeta)=\frac{s\sin\tau}{\tau\cos\tau-i\zeta\sin\tau},$$

where $\tau=\sqrt{\zeta^2+|s|^2}$. Thus $L(0)=e^{i\theta}\tan r$.

Although a direct relation of the range of $\mathfrak{F}$ and the coefficient S is difficult to obtain, the following proposition gives it for restricted S.

Proposition 3.5.1. *Suppose $s(x)$, the entry of $S(x)\in\mathcal{S}^p_{r,n}$ has following properties.*

(a) $s(x)=r(x)e^{i\theta}$, where θ is a constant and $r(x)$ is real valued function.

(b) $\int_0^\infty r(x)dx=n\pi$ for some integer n.

Then $S(x)$ can be transformed to $Q(x)\in\mathcal{Q}^p_{r,n}$ with the same scattering data via $\mathfrak{F}^{-1}$.

Proof. Consider $\dfrac{dy}{dx}=Ay$ for a constant matrix $A=\begin{pmatrix} 0 & h^* \\ -h & 0 \end{pmatrix}$. The fundamental solution $U(h;x,x_0)$ is given by

$$U(h;x,x_0)=\frac{1}{2}\begin{pmatrix} e^{i|h|(x-x_0)}+e^{-i|h|(x-x_0)} & -i\frac{|h|}{h}\left(e^{i|h|(x-x_0)}-e^{-i|h|(x-x_0)}\right) \\ i\frac{|h|}{h^*}\left(e^{i|h|(x-x_0)}-e^{-i|h|(x-x_0)}\right) & e^{i|h|(x-x_0)}+e^{-i|h|(x-x_0)} \end{pmatrix}.$$

Thus, for a step coefficient

$$s(x)=\sum_{k=1}^{n}h_k\chi_{[x_{k-1},x_k]},\quad x_0=0,x_n=X,$$

the limit of a solution $F^\dagger$ to (3.20) given by

$$F^\dagger_\infty = \prod_{k=1}^{n} U(h_k; x_k, x_{k-1}) \cdot I.$$

Here $\prod_{k=1}^{n+1} A_k = A_{n+1}\prod_{k=1}^{n} A_k$. To get a closed form of $\prod_{k=1}^{n} U(h_k; x_k, x_{k-1})$, we assume that $x_k - x_{k-1} = \delta$ for all k, and $h_k = r_k e^{i\theta}$. Then

$$U(h_k; x_k, x_{k-1}) = \begin{pmatrix} \cos(r_k\delta) & e^{-i\theta}\sin(r_k\delta) \\ -e^{i\theta}\sin(r_k\delta) & \cos(r_k\delta) \end{pmatrix}.$$

Denote $c_k = \cos(r_k\delta)$ and $s_k = \sin(r_k\delta)$ for a moment. Since

$$U(h_{k+1}; x_{k+1}, x_k)U(h_k; x_k, x_{k-1}) = \begin{pmatrix} c_k c_{k+1} - s_k s_{k+1} & e^{-i\theta}(c_{k+1}s_k + s_{k+1}c_k) \\ -e^{i\theta}(c_{k+1}s_k + s_{k+1}c_k) & c_k c_{k+1} - s_k s_{k+1} \end{pmatrix},$$

by induction

$$\prod_{k=1}^{n} U(h_k; x_k, x_{k-1}) = \begin{pmatrix} \cos(\delta\sum r_k) & e^{-i\theta}\sin(\delta\sum r_k) \\ -e^{i\theta}\sin(\delta\sum r_k) & \cos(\delta\sum r_k) \end{pmatrix}.$$

Since the set of step functions is a dense subset of L^p and $\sin x, \cos x$ are continuous functions,

$$\begin{aligned} F^\dagger(X) &= \lim_{n\to\infty}\prod_{k=1}^{n} U(h_k, x_{k-1} + X/n, x_{k-1}), \\ &= \begin{pmatrix} \cos(\int_0^X r(x)dx) & e^{-i\theta}\sin(\int_0^X r(x)dx) \\ -e^{i\theta}\sin(\int_0^X r(x)dx) & \cos(\int_0^X r(x)dx) \end{pmatrix}. \end{aligned}$$

Now send $X \to \infty$, in which case $Q(X) \to \Lambda$ provided $\sin(\int_0^\infty r(x)dx) = 0$ or $\int_0^\infty r(x)dx = n\pi$. $\square$

Note that this argument can not be adapted to a general $S(x)$, since it looks quite difficult to find closed form of $\prod U(h_k, x_k, x_{k-1})$ for a general $S(x)$.

Since $\mathfrak{F}(\mathcal{Q})$ is a strict subset of $\mathcal{S}$, it is natural to extend $\mathfrak{F}$ to a superset of $\mathcal{Q}$ to make it a one to one correspondence with $\mathcal{S}$. One of the approaches is to consider the set of step-like coefficients, $\mathcal{Q}^s$ defined in (3.37).[3]

[3] We also drop the subscripts p, r, n from $\mathcal{Q}^{s,p}_{r,n}$.

We showed $\mathfrak{F}_s$ is an injective map from $\mathcal{Q}^s$ to $\mathcal{S}$ with F_0 given by (3.36) in Theorem 3.4.1. The transformed scattering data is given by (3.41)-(3.43). That is, the knowledge of $Q(0)$ and the scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^N$ give the scattering data for the ZSSP with coefficient $\mathfrak{F}_s Q$. Now we drop subscript s from $\mathfrak{F}_s$, since it can be considered as an extension of the transformation $\mathfrak{F}$.

Now we seek an inverse transformation of $\mathfrak{F}$ from $\mathcal{S}$ to $\mathcal{Q}^s$. As stated in Theorem 3.4.1, for $S \in \mathfrak{F}\mathcal{Q}^s$

$$F_x^\dagger = \Lambda S F^\dagger, \quad F^\dagger(0) = F_0^\dagger, \tag{3.49}$$

is uniquely solvable for any $F_0^\dagger = \begin{pmatrix} f_0^* & g_0^* \\ -g_0 & f_0 \end{pmatrix}$, and then Q is given by (3.21),

$$Q = F \Lambda F^\dagger. \tag{3.50}$$

Now assume that $S \in \mathcal{S}$. Q can be constructed by the same procedure, but we have to find a condition for $F_0^\dagger$ so that $Q \in \mathcal{Q}^s$ or equivalently $\lim_{x\to\infty} F(x) = F_\infty$ is a diagonal matrix. Let $\breve{\mathcal{T}} = \begin{pmatrix} \breve{a}(\zeta) & -\breve{b}^*(\zeta^*) \\ \breve{b}(\zeta) & \breve{a}^*(\zeta^*) \end{pmatrix}$ be the transition matrix, and likewise $\breve{J}^\pm$, $\breve{L}$ etc. for the ZSSP with S.

Since $\breve{J}^-(x,0)$ solves (3.49) with initial condition $\breve{J}^-(0,0) = I$, the solution $F^\dagger$ of (3.49) should be

$$F^\dagger(x) = \breve{J}^-(x,0) F_0^\dagger.$$

From (3.48)

$$F_\infty = F_0 \breve{\mathcal{T}}(0).$$

Then, $\mathcal{D}^c F_\infty = 0$ if and only if

$$\breve{a}(0) g_0 + \breve{b}(0) f_0^* = 0, \quad \text{or } g_0 + \breve{L}(0) f_0^* = 0. \tag{3.51}$$

Thus for any unitary matrix F_0 satisfying the condition (3.51), $\mathfrak{F}^{-1}$ is well defined from $\mathcal{S}$ to $\mathcal{Q}^s$. In particular,

$$Q(0) = F_0 \Lambda F_0^\dagger.$$

By the same argument in Section 3.4, one can show that the transition matrix $\mathcal{T}$ for the LLSP with coefficient $\mathfrak{F}^{-1}S$ is given by

$$\mathcal{T} = F_0 \breve{\mathcal{T}} F_\infty^\dagger.$$

So, the corresponding scattering data $\{L(\zeta), \zeta_j, C_{r,n}\}_{n=1}^N$ is also given by

$$\begin{aligned} L(\zeta) &= \frac{\breve{a}(\zeta) g_0 + \breve{b}(\zeta) f_0^*}{\breve{a}(\zeta) f_0 - \breve{b}(\zeta) g_0^*} \\ &= \frac{g_0 + \breve{L}(\zeta) f_0^*}{f_0 - \breve{L}(\zeta) g_0^*}, \end{aligned} \tag{3.52}$$

$$\zeta_n = \breve{\zeta}_n, \tag{3.53}$$

$$C_{r,n} = \breve{C}_{r,n}. \tag{3.54}$$

Similarly to the map $\mathfrak{F}$ from $\mathcal{Q}^s$ to $\mathcal{S}$, $\mathfrak{F}^{-1}S$ depends F_0 up to constant multiplication. We can choose specific F_0 as did for $\mathfrak{F}$ by (3.36). Since $\det F_0 = 1$, (3.51) yields

$$|f_0|^2 (1 + |\breve{L}(0)|^2) = 1.$$

One can choose f_0 as

$$f_0 = \frac{1}{\sqrt{1 + |\breve{L}(0)|^2}}. \tag{3.55}$$

Since f_0 defined above is real value, we can rewrite (3.52) by aid of (3.51) and (3.55) as

$$L(\zeta) = \frac{-\breve{L}(0) + \breve{L}(\zeta)}{1 + \breve{L}^*(0)\breve{L}(\zeta)}. \tag{3.56}$$

Theorem 3.5.2. *The inverse transformation $\mathfrak{F}^{-1}$ from $\mathcal{S}$ to $\mathcal{Q}^s$ is well defined by (3.49), (3.51) and (3.50). The transformed Q and the scattering data depend on F_0 or f_0. If f_0 is given by (3.55), then Q is uniquely determined and the corresponding left reflection coefficient is given by (3.56).*

Another method to define a transformation from $\mathcal{S}$ to $\mathcal{Q}$ is to consider the inverse transformation $\mathfrak{F}_a^{-1}$. In this case, the scattering data is invariant through the transformation.

Theorem 3.5.3. *Let $S \in \mathcal{S}$ be a coefficient of the ZSSP. Suppose that $\kappa_0 i$ is a pure imaginary eigenvalue corresponding to S. Then for any $\beta \in \mathbb{R}^-$ such that $-\beta \neq \kappa_0$ or for any $\beta \in \mathbb{R}^+$ such that $b(i\beta) = 0$, there uniquely exists $Q \in \mathcal{Q}$, coefficient in the ALLSP, satisfying*

$$F_x = -F\Lambda S - \beta\Lambda\mathcal{D}^c(F\Lambda F^\dagger)F, \quad F(0) = I, \tag{3.57}$$

$$Q = F\Lambda F^\dagger, \tag{3.58}$$

and which generates the same scattering data as S.

Proof. The main point of this theorem is to preserve the scattering data through the transformation in contrast to $\mathfrak{F}^{-1}$. For the proof, we have to show that F, the solution to (3.57), converges to

$$F_\infty = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}, \quad |\alpha| = 1. \tag{3.59}$$

Then the scattering data is unchanged as we discussed earlier. As shown in Section 3.3, (3.57) is uniquely solvable for any $S \in \mathcal{S}$ and $\beta \in \mathbb{R}$, and $\lim_{x\to\infty} F(x)$ exists. In order to show that Q defined by (3.58) is in $\mathcal{Q}$, we have to investigate the structure of F.

Since $\Lambda\mathcal{D}^c A = -\mathcal{D}^c A\Lambda$ for any matrix A,

$$\begin{aligned} F^\dagger F_x + F_x^\dagger F &= (-\Lambda S - \beta F^\dagger \Lambda\mathcal{D}^c(F\Lambda F^\dagger)F) + (-S\Lambda - \beta F^\dagger\mathcal{D}^c(F\Lambda F^\dagger)\Lambda F), \\ &= 0. \end{aligned}$$

With the initial condition $F(0) = I$, it follows that $F^\dagger F = I$ for all x. Furthermore $\mathrm{trace}(\Lambda S) = \mathrm{trace}(\Lambda\mathcal{D}^c(F\Lambda F^\dagger)) = 0$ implies $\det F = 1$ by Liouville's formula. Thus,

$$F(x) = \begin{pmatrix} f(x) & -g^*(x) \\ g(x) & f^*(x) \end{pmatrix}, \quad |f(x)|^2 + |g(x)|^2 = 1,$$

and Q has the form of (3.2) with $Q(0) = \Lambda$.

Next, we need to check the asymptotic behavior of F. Let

$$\lim_{x\to\infty} F(x) = \begin{pmatrix} f_\infty & -g_\infty^* \\ g_\infty & f_\infty^* \end{pmatrix}.$$

Rewrite (3.57) componentwise,

$$f^*_x = s^*(-g) - 2\beta f^*|g|^2, \quad f^*(0) = 1, \tag{3.60}$$

$$(-g)_x = -sf^* + 2\beta|f|^2(-g), \quad -g(0) = 0. \tag{3.61}$$

Define[4] $v = \exp(-\beta \int_0^x |f|^2 - |g|^2) \begin{pmatrix} f^* \\ -g \end{pmatrix}$. Then

$$v_x = i(i\beta)\Lambda v + \Lambda S v, \quad v(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

That is, v solves the ZSSP for $\zeta = i\beta$. Moreover, from the initial condition of v,

$$v(x) = \nu(x, i\beta).$$

Thus, for $\mathcal{T} = \begin{pmatrix} a(\zeta) & -b^*(\zeta^*) \\ b(\zeta) & a^*(\zeta^*) \end{pmatrix}$

$$v(x) = a^*(-i\beta)\mu(x, i\beta) - b(i\beta)\overline{\mu}(x, i\beta). \tag{3.62}$$

Note that $b(i\beta)$ exists for all $\beta \in \mathbb{R}$ without assuming that $S(x)$ decays sufficiently rapidly, since it is supported on the half line. Now send $x \to \infty$. Then

$$f^*_\infty = \lim_{x\to\infty} a^*(-i\beta) \exp \beta(-x + \int_0^x |f|^2 - |g|^2), \tag{3.63}$$

$$-g_\infty = \lim_{x\to\infty} -b(i\beta) \exp \beta(x + \int_0^x |f|^2 - |g|^2). \tag{3.64}$$

Suppose $\beta > 0$. Since the both limits exist, from (3.64) $|g_\infty| = 1$ and $|f_\infty| = 0$ or $b(i\beta) = 0$. If $b(i\beta) = 0$ then $g_\infty = 0$ and $|f_\infty| = 1$. Now we assume that $\beta < 0$. Then $|g_\infty| = 0, |f_\infty| = 1$ from (3.63) provided $a(-i\beta) \neq 0$, that is, if $-i\beta$ is not an eigenvalue. In case of $a(-i\beta) = 0$, $f_\infty = 0$ from (3.63), thus $|g_\infty| = 1$.

Theorem 3.5.3 follows from that $Q = F\Lambda F^\dagger$ and the scattering data is unchanged only if $|f_\infty| = 1, |g_\infty| = 0$ as mentioned in (3.59) and in Section 3.2. $\square$

[4] One can find v from the Prüfer transformation of (3.60),(3.61)

As a corollary, we know $\mathfrak{F}_a$ is not a one to one map. Indeed, there are infinitely many (Q, β) exist, which transform to a single S. Also $\mathfrak{F}_a(\mathcal{Q}^p_{r,n}, \beta_0) \subsetneq \mathcal{S}^p_{r,n}$ in general. Consider $S \in \mathcal{S}^p_{r,n}$ which has bound state at $\zeta_0 = -i\beta_0$ for $\beta_0 < 0$. Then for the corresponding Q given by (3.58), $\lim_{x\to\infty} Q(x) = -\Lambda$, so $Q \notin \mathcal{Q}^p_{r,n}$.

We remark that under this inverse transformation $\mathfrak{F}_a^{-1}$, the support of coefficient may be changed. Although S has a compact support, the transformed $Q - \Lambda$ may be infinitely supported. On the other hand, the direct transformation $\mathfrak{F}_a$ preserved the support. That is, if $Q - \Lambda$ has supported on $[0, X]$, then the support of transformed S is subset of $[0, X]$.

CHAPTER 4. Scattering problems without support restriction

4.1 Introduction

Up to now we have discussed problems with compactly supported or semi-infinite coefficients. In this chapter, we remove the support restriction by combining two half line problems. This approach was first investigated by Aktosun and Sacks for the Schrödinger equation ([10]). They split the whole line problem into two half line problems assuming no bound states. From a numerical point of view, this method may reduce the computational cost from $O(N^3)$ to $O(N^2)$, where N is a number of mesh points.

Here, we apply their idea to the ZSSP. We can extract the left and right reflection data for the right and left half line problems respectively. After that, we can restore the half line coefficients even if they have bound states from Theorem 2.5.1 and the Darboux transformation.

In the LLSP, we are also able to apply the splitting method by using step-like coefficients which are discussed in Section 3.4. Then the transformation $\mathfrak{F}$ may be applied to the each half line problem.

4.2 Splitting method for the ZSSP

In this section we discuss how scattering data for half line problems are extracted from the given (right) scattering data of a whole line problem. Recall the ZSSP,

$$\frac{\partial \phi}{\partial x} = i\zeta\Lambda\phi + \Lambda S\phi,$$

where,

$$S \in \mathcal{S}_n^p = \{S : S \in L_n^p(\mathbb{R}) \cap L^1(\mathbb{R})\}.$$

Note that $S \in \mathcal{S}_n^p$ can be represented by the sum of S_1 and S_2, which are

$$S_1(x) = S(x)\chi_{(-\infty,0)}, \quad S_2(x) = S(x)\chi_{(0,\infty)}.$$

Since $S_1 \in \mathcal{S}_{l,n}^p$ and $S_2 \in \mathcal{S}_{r,n}^p$,

$$\mathcal{S}_n^p = \mathcal{S}_{l,n}^p + \mathcal{S}_{r,n}^p. \tag{4.1}$$

Here,

$$\begin{aligned} \mathcal{S}_{l,n}^p &= \{S : S \in L_n^p(\mathbb{R}^-) \cap L^1(\mathbb{R}^-)\}, & (4.2)\\ \mathcal{S}_{r,n}^p &= \{S : S \in L_n^p(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)\}. & (4.3) \end{aligned}$$

Note that the definitions of $\mathcal{S}_{l,n}^p$ and $\mathcal{S}_{r,n}^p$ are not exactly same as $\mathcal{S}_{l,n}^p$ and $\mathcal{S}_{r,n}^p$ shown in the previous chapter. Here we remove the generic condition. Again, we drop the subscripts n, p for each $\mathcal{S}_n^p, \mathcal{S}_{r,n}^p$ and $\mathcal{S}_{l,n}^p$ for notational simplicity.

Let $J^\pm, J_1^\pm, J_2^\pm$ be the Jost solutions for S, S_1, S_2 respectively. The transition matrices $\mathcal{T}, \mathcal{T}_j$ $(j = 1, 2)$ are also defined by

$$J^+ = J^-\mathcal{T}, \quad J_j^+ = J_j^-\mathcal{T}_j, \quad j = 1, 2 \tag{4.4}$$

and likewise the scattering data R_1, L_1 etc. is defined for each half line coefficient . Then we can make the following statement.

Theorem 4.2.1. *For $S \in \mathcal{S}$ and $x \in \mathbb{R}, \zeta \in \mathbb{R}$,*

(1) $J^+ = J_1^+ \exp(-i\zeta\Lambda x)J_2^+$,

(2) $J^- = J_2^- \exp(-i\zeta\Lambda x)J_1^-$,

(3) $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2$.

Proof. (1) If $x > 0$, then $J_1^+ = \exp(i\zeta\Lambda x)$, so $J_1^+ \exp(-i\zeta\Lambda x)J_2^+ = J_2^+$. For $x < 0$,

$$\begin{aligned} J_1^+ \exp(-i\zeta\Lambda x)J_2^+ &= J_1^+ \exp(-i\zeta\Lambda x)J_2^-\mathcal{T}_2 \\ &= J_1^+\mathcal{T}_2. \end{aligned}$$

Since $J_1^+ \exp(-i\zeta\Lambda x) J_2^+$ is continuous at $x = 0$, and J_2^+ and $J_1^+ \mathcal{T}_2$ solve (4.1) on the right and left half lines respectively, $J_1^+ \exp(-i\zeta\Lambda x) J_2^+$ is a solution of (4.1). Moreover, for $x > 0$

$$J_1^+ \exp(-i\zeta\Lambda x) J_2^+ = J_2^+ \to \exp(i\zeta\Lambda x), \quad \text{as } x \to \infty.$$

The uniqueness of the Jost solutions shows that $J_1^+ \exp(-i\Lambda\zeta x) J_2^+$ is the right Jost solution for S.

(2) Similarly, for $x > 0$

$$\begin{aligned} J_2^- \exp(-i\zeta\Lambda x) J_1^- &= J_2^- \exp(-i\zeta\Lambda x) J_1^+ \mathcal{T}_1^{-1} \\ &= J_2^- \mathcal{T}_1^{-1}. \end{aligned}$$

If $x < 0$

$$J_2^- \exp(-i\zeta\Lambda x) J_1^- = J_1^-.$$

Hence, $J^- = J_2^- \exp(-i\zeta\Lambda x) J_1^-$. Note that if ϕ is a solution of (4.1), then $\phi A(\zeta)$ also solves (4.1) for any matrix $A(\zeta)$.

(3) If $x > 0$, from part (b)

$$J^- = J_2^- \exp(-i\zeta\Lambda x) J_1^- = J_2^- (J_1^+)^{-1} J_1^- = J_2^- (J_1^- \mathcal{T}_1)^{-1} J_1^- = J_2^- \mathcal{T}_1^{-1}.$$

The uniqueness of (4.1) implies $J^+ = J_2^+$ for $x > 0$, thus

$$\begin{aligned} J^+ &= J^- \mathcal{T} \\ J_2^+ &= J_2^- \mathcal{T}_1^{-1} \mathcal{T} \\ J_2^- \mathcal{T}_2 &= J_2^- \mathcal{T}_1^{-1} \mathcal{T} \\ \mathcal{T} &= \mathcal{T}_1 \mathcal{T}_2. \end{aligned}$$

Similarly we can show this for $x < 0$.

□

By induction, we have following corollary.

Corollary 4.2.2. *Let* $S_j = S\chi_{(x_j,x_j+1)}$ *be the fragments of* S*, where* $x_0 = -\infty, x_{N+1} = \infty$*. Then the transition matrix* $\mathcal{T}$ *for* S *is given by*

$$\mathcal{T} = \mathcal{T}_0 \cdots \mathcal{T}_{N-1}\mathcal{T}_N,$$

where $\mathcal{T}_j$ *is the transition matrix for* S_j*.*

The same result for Schrödinger equation can be found in [6].

We may now state a procedure for direct determination of the two half line reflection coefficients R_1, L_2 from the right hand scattering data $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^{N}$ which is analogous to that given for the Schrödinger equation in [10].

Recall the GLM equation (2.17),

$$\overline{K}(x,y) + \tilde{M}(x+y)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^\infty K(x,z)\tilde{M}(z+y)dz = 0, \quad x < y, \tag{4.5}$$

where,

$$K = \begin{pmatrix} K^{(1)} \\ K^{(2)} \end{pmatrix}, \quad \overline{K} = -i\sigma_y K^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K^*, \tag{4.6}$$

$$\tilde{M}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} R(\zeta)e^{i\zeta x}d\zeta - i\sum_{n=1}^{N} C_{l,n}e^{i\zeta_n x}. \tag{4.7}$$

Introducing the real and imaginary parts of $K(0,y)$ and $\tilde{M}(y)$ as $K(0,y) = K_R(y) + K_I(y)i, \tilde{M}(y) = M_R(y) + M_I(y)i$, we may rewrite (4.5) at $x = 0$ as

$$-i\sigma_y K_R(y) + \lambda M_R(y) + \int_0^\infty K_R(z)M_R(z+y) - K_I(z)M_I(z+y)dz = 0, \tag{4.8}$$

$$i\sigma_y K_I(y) + \lambda M_I(y) + \int_0^\infty K_R(z)M_I(z+y) + K_I(z)M_R(z+y)dz = 0, \tag{4.9}$$

where $\lambda = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The above system of integral equations can be solved for K_R, K_I by several methods due to its symmetric structure. Once the solution to (4.8),(4.9) is given, we subsequently can obtain $J^+(0,\zeta)$ by setting $x = 0$ in (2.2), that is

$$J^+(0,\zeta) = I + [\mathbf{K} \quad -i\sigma_y\mathbf{K}^*]. \tag{4.10}$$

Here,

$$\mathbf{K}(\zeta) = \int_0^\infty K(0,y)e^{i\zeta y}dy.$$

On the other hand,

$$I = J_2^-(0,\zeta) = J_2^+(0,\zeta)\mathcal{T}_2^{-1}(\zeta) = J^+(0,\zeta)\mathcal{T}_2^{-1}(\zeta), \tag{4.11}$$

$$\mathcal{T}_2(\zeta) = J^+(0,\zeta), \tag{4.12}$$

thus $\mathcal{T}_2(\zeta)$, and in particular

$$L_2(\zeta) = \frac{b_2(\zeta)}{a_2(\zeta)} \tag{4.13}$$

is obtained directly.

Finally Theorem 4.2.1 gives

$$\mathcal{T}_1 = \mathcal{T}\mathcal{T}_2^{-1}, \tag{4.14}$$

or

$$\begin{pmatrix} a_1 & -b_1^* \\ b_1 & a_1^* \end{pmatrix} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} a_2^* & b_2^* \\ -b_2 & a_2 \end{pmatrix}. \tag{4.15}$$

Thus,

$$R_1 = \frac{b_1^*}{a_1} = \frac{-b_2^* a + a_2 b^*}{a_2^* a + b_2 b^*} = \frac{-b_2^* + a_2 R}{a_2^* + b_2 R}. \tag{4.16}$$

Algorithm 4.2.3.

1. *From the given scattering data* $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$ *solve the system of equations* (4.8), (4.9) *for* K_R, K_I.

2. *Compute* $\mathcal{T}_2(\zeta)$ *from* (4.10) *and* (4.12).

3. *Extract* $L_2(\zeta)$ *from* $\mathcal{T}_2(\zeta)$, *and* $R_1(\zeta)$ *from* (4.16)

Theorem 4.2.4. *The half line reflection coefficients* R_1, L_2 *may be uniquely recovered from the scattering data* $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$ *from Algorithm 4.2.3.*

Figure 4.1 shows a numerical example of $s(x)$ reconstructed on $[-0.5, 0.5]$ from the scattering data $\{R(\zeta), \zeta_1, C_{l,1}\}$. The right reflection coefficient $R(\zeta)$ is again generated by an ODE solver, the eigenvalue is computed by minimizing $|a(\zeta)|^2$ in the upper half plane, and a simple

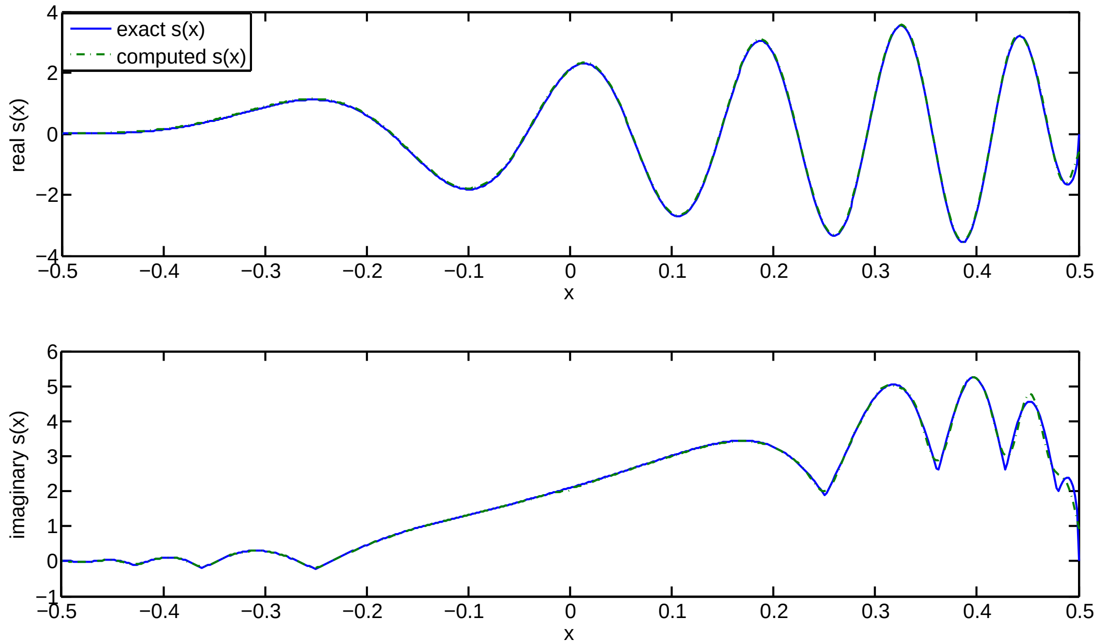

Figure 4.1 Splitting method for the ZSSP

finite difference method is used for the normalizing constant. We extract $\{R_1(\zeta), L_2(\zeta)\}$ by Algorithm 4.2.3. The system of integral equations (4.8), (4.9) is solved via a simple discretization by the trapezoidal rule.

Since the each half line coefficients S_j might be considered as discontinuous coefficients, $R_1(\zeta), L_2(\zeta) \sim O(1/\zeta)$ from (2.112). Thus, we use the fast Fourier transformation with padding by C/ζ instead of padding by zero. Here, we use averages of $R_1(\zeta), L_2(\zeta)$ over a certain range to determine C. It turns out $s_2(x)$ has one bound state by checking $\widehat{L}_2(t)$ for $t < 0$. So we have to use the Darboux transformation. To recover $s_1(x)$ from $R_1(\zeta)$, we convert this problem to a left scattering problem by using a symmetry. At the splitting point, $x = 0$, $s(x)$ should be computed by a sum of $s_1(0^-)$ and $s_2(0^+)$.

4.3 Splitting method for the LLSP

We have developed several transformations from the LLSP and the ZSSP on the half line in Chapter 3, so that we can solve the LLSP from the converted ZSSP. In this section, we remove the restriction on the support. As the ZSSP discussed in the previous section, the whole line LLSP can be split into two half line problems with step-like coefficients. Then the transformation $\mathfrak{F}$, more precisely, $\mathfrak{F}_s$ may be applied.

Define

$$\mathcal{Q}_n^p = \{Q : Q - \Lambda \in H_n^p(\mathbb{R}) \cap H^1(\mathbb{R})\}.$$

For $Q \in \mathcal{Q}_n^p$, we can not simply split as $Q = Q_1 + Q_2$ for coefficients Q_1, Q_2 which are supported on the half lines as done in the ZSSP, since Q_1, Q_2 should be continuous, or have continuous extensions on the whole real line. Thus, we define function spaces $\mathcal{Q}_{l,n}^{s,p}$ and $\mathcal{Q}_{r,n}^{s,p}$ of step-like coefficients as follows.

$$\begin{aligned}
\mathcal{Q}_{l,n}^{s,p} &:= \{Q : Q - \Lambda \in H_n^p(\mathbb{R}^-) \cap H^1(\mathbb{R}^-),\ Q(x) = Q(0) \text{ for } x \geq 0\},\\
\mathcal{Q}_{r,n}^{s,p} &:= \{Q : Q - \Lambda \in H_n^p(\mathbb{R}^+) \cap H^1(\mathbb{R}^+),\ Q(x) = Q(0) \text{ for } x \leq 0\}.
\end{aligned}$$

Note that we remove the generic condition from the definitions of $\mathcal{Q}_{l,n}^{s,p}, \mathcal{Q}_{r,n}^{s,p}$ similarly to $\mathcal{S}_{l,n}^p, \mathcal{S}_{r,n}^p$ in Section 4.2. Since $Q_1 \in \mathcal{Q}_{l,n}^{s,p}, Q_2 \in \mathcal{Q}_{r,n}^{s,p}$ are extended by $Q_j(0)(j = 1, 2)$ for $x > 0$ and $x < 0$ respectively, Q_j is differentiable on the whole line. For any $Q \in \mathcal{Q}_n^p$, there are unique $Q_1 \in \mathcal{Q}_{l,n}^{s,p}$ and $Q_2 \in \mathcal{Q}_{r,n}^{s,p}$ such that

$$Q(x) = Q_1(x)\chi_{(-\infty,0]} + Q_2(x)\chi_{(0,\infty)}.$$

We drop the subscripts s, n, p for notational simplicity.

The following theorem is an analogy to Theorem 4.2.1.

Theorem 4.3.1. *Let $J^\pm, J_j^\pm (j = 1, 2)$ be the Jost solution for $Q \in \mathcal{Q}, Q_1 \in \mathcal{Q}_l, Q_2 \in \mathcal{Q}_r$ respectively and $\mathcal{T}, \mathcal{T}_j$ be the corresponding transition matrixes. Then for $x \in \mathbb{R}, \zeta \in \mathbb{R}$*

(1) $J^+ = J_1^+ \exp(-i\zeta Q_0 x) J_2^+$,

(2) $J^- = J_2^- \exp(-i\zeta Q_0 x) J_1^-$,

(3) $\mathcal{T} = \mathcal{T}_1\mathcal{T}_2$,

where $Q_0 = Q(0)$.

One can easily prove this theorem by the similar way to the proof of Theorem 4.2.1 together with properties of the Jost solutions,

$$J_1^+ = \exp(i\zeta Q_0 x), \quad x > 0, \qquad J_2^- = \exp(i\zeta Q_0 x), \quad x < 0.$$

As we showed in Section 3.4, Q_2 in the LLSP can be transformed to $S_2 \in \mathcal{S}_r$ in the ZSSP. The corresponding transition matrix $\tilde{\mathcal{T}}_2$ for S_2 is given by (3.40),

$$\tilde{\mathcal{T}}_2 = F_0^\dagger \mathcal{T}_2 F_\infty, \tag{4.17}$$

where F_0 satisfies (3.29) and $F_\infty = \lim_{x\to\infty} F(x)$ for the solution $F(x)$ to (3.38).

Similarly to $\tilde{\mathcal{T}}_2$, one can define $\tilde{\mathcal{T}}_1$. Let $F(x)$ be the solution of (3.38) for $x < 0$, and

$$F_{-\infty} = \lim_{x\to-\infty} F(x).$$

Multiply $F^\dagger(x)$ and F_0 to the left and right of each side of $J_1^+ = J_1^- \mathcal{T}_1$ respectively. Then

$$F^\dagger J_1^+ F_0 = F^\dagger J_1^- \mathcal{T}_1 F_0,$$

or

$$F^\dagger J_1^+ F_0 = F^\dagger J_1^- F_{-\infty} F_{-\infty}^\dagger \mathcal{T}_1 F_0.$$

It is not difficult to show that $F^\dagger J_1^+ F_0$ and $F^\dagger J_1^- F_{-\infty}$ are the right and left Jost solutions for the ZSSP with S_1 by the same argument in Section 3.4. Thus the transition matrix $\tilde{\mathcal{T}}_1$ for S_1 is given by

$$\tilde{\mathcal{T}}_1 = F_{-\infty}^\dagger \mathcal{T}_1 F_0.$$

Together with (4.17) and Theorem 4.2.1, the transition matrix $\tilde{\mathcal{T}}$ for $S = S_1 + S_2$ can be written by

$$\tilde{\mathcal{T}} = \tilde{\mathcal{T}}_1 \tilde{\mathcal{T}}_2 = F_{-\infty}^\dagger \mathcal{T}_1 F_0 F_0^\dagger \mathcal{T}_2 F_\infty.$$

Due to Theorem 4.3.1 and the property of F_0, we have

$$\tilde{\mathcal{T}} = F_{-\infty}^\dagger \mathcal{T} F_\infty.$$

Even if $F_\infty, F_{-\infty}$ are diagonal matrices, the scattering data, generally, is not invariant. Indeed, $\tilde{R}(\zeta)$ is given by

$$\tilde{R}(\zeta) = \frac{f_\infty^*}{f_\infty} R(\zeta), \quad \text{for } F_\infty = \begin{pmatrix} f_\infty & 0 \\ 0 & f_\infty^* \end{pmatrix}.$$

The specific choice of initial condition may solve this problem. Let $G(x)$ solve (3.38) with $G(0) = G_0$ such that

$$G_0 = F_0 F_\infty^\dagger. \tag{4.18}$$

Then $G(x) = F(x) F_\infty^\dagger$, and

$$\tilde{\mathcal{T}} = F_\infty F_{-\infty}^\dagger \mathcal{T}. \tag{4.19}$$

Since $F_\infty F_{-\infty}^\dagger$ is a unitary diagonal matrix, the right scattering data for the LLSP with Q coincides with one for the ZSSP with S.

Theorem 4.3.2. *For any $Q \in \mathcal{Q}$, there is a matrix G_0 which makes the transform $\mathfrak{F}$ well defined and the right scattering data invariant under $\mathfrak{F}$. With this G_0, $\mathfrak{F}$ is a one to one map from $\mathcal{Q}$ to $\mathcal{S}$. Similarly, there uniquely exists G_0' which makes the left scattering data invariant under the transformation.*

We remark that there exist two G_0, which are characterized by $G_\infty = \lim_{x\to\infty} G(x) = \pm I$. From the construction of G_0 in (4.18), obviously $G_\infty = I$. Suppose that $\tilde{G}$ solving (3.38) with the initial condition $\tilde{G} = \tilde{G}_0$ converges to I as $x \to \infty$. Since $QG = G\Lambda$ and $Q\tilde{G} = \tilde{G}\Lambda$, for some unitary diagonal matrix C

$$\tilde{G} = GC.$$

Let $E = G - \tilde{G}$. Then

$$E_x = \frac{1}{2} Q_x Q E, \quad \lim_{x\to\infty} E(x) = 0.$$

From Liouville's formula, $\det E(x) = 0$. This implies $\det(I - C) = 0$ or $G = \tilde{G}$. Similarly, one can show that there is a unique G_0 such that $G_\infty = -I$, which is given by

$$G_0 = -F_0 F_\infty^\dagger.$$

In this procedure, the knowledge of F_0, which is not given in the inverse problem, looks to be essential to solve the LLSP via the transformation $\mathfrak{F}$. The following algorithm, however, shows how to solve the LLSP on the whole line by the splitting method of the transformed ZSSP without a priori information of F_0.

Algorithm 4.3.3.

1. *The given scattering data $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^N$ for the LLSP with $Q \in \mathcal{Q}$ may be considered as the scattering data for the ZSSP with $S \in \mathcal{S}$ by Theorem 4.3.2.*

2. *Extract the right and left reflection coefficients R_1 and $L_2(\mathcal{T}_2)$ for S_1 and S_2 respectively as described in Algorithm 4.2.3.*

3. *Solve the ZSSP for S_1 and S_2 from R_1 and L_2 respectively by the methods developed in Chapter 2.*

4. *The coefficient Q or Q_j $(j = 1, 2)$ of the LLSP can be recovered from (3.50)*

$$Q_j = G_j \Lambda G_j^\dagger.$$

Here, G_j is a solution of (3.49)

$$G_{j,x}^\dagger = \Lambda S_j G_j^\dagger, \quad G_j^\dagger(0) = G_0^\dagger.$$

The initial condition G_0 is given by

$$G_0 = \mathcal{T}_2^\dagger(0). \tag{4.20}$$

Indeed, from Section 3.5 (see (3.48))

$$\lim_{x\to\infty} G_2(x) = G_0 \mathcal{T}_2(0).$$

The condition $\lim_{x\to\infty} G_2(x) = I$ implies (4.20).

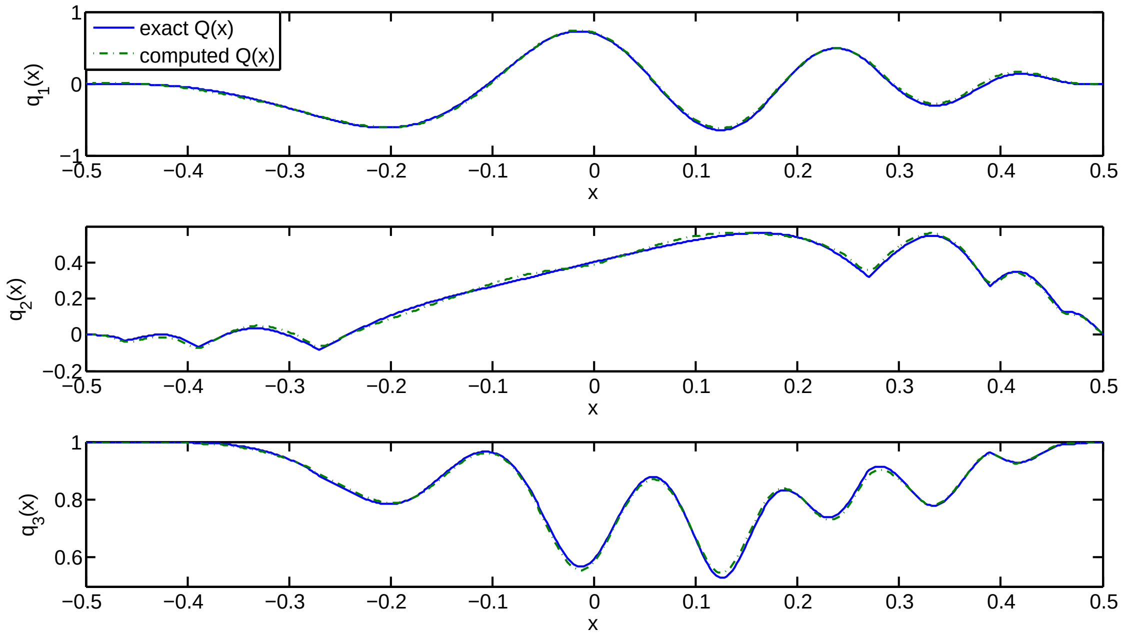

Figure 4.2 Splitting method for the LLSP

Figure 4.2 shows a numerical example of $Q(x)$ reconstructed via Algorithm 4.3.3. Similarly to the example given in Section 4.2, $Q_2(x)$ has one bound state and we convert the right scattering problem, the LLSP with $Q_1(x)$, to a left scattering problem. At the final step, $Q(0)$ is restored from $Q_1(0^-)$ or $Q_2(0^+)$. Since $Q_j(x)(j = 1, 2)$ is continuous at $x = 0$, we do not need to use the sum of $Q_1(0^-)$ and $Q_2(0^+)$ as we did in the ZSSP case.

CHAPTER 5. Uniqueness and non-uniqueness for the Landau-Lifschitz scattering problem

5.1 Introduction

In this chapter we discuss the LLSP with a discontinuous coefficient . If $Q(x)$ is smooth enough then the problem is solvable via the transformation $\mathfrak{F}$ that we discussed in Chapter 3 and 4. The main equation for $\mathfrak{F}$ is (3.7),

$$F_x = \frac{1}{2} Q_x Q F. \tag{5.1}$$

Suppose that Q has a jump at a point x_0. Then (5.1) is not defined even in obvious distributional sense, since $Q_x Q$ is a product of distributions. To avoid this difficulty, we need a direct approach to solve the LLSP.

With a discontinuous coefficient, we have an example which shows that the inverse problem for the LLSP is not unique. The counter example is constructed from a piecewise constant coefficient, which is introduced in Section 5.2. Also, Uniqueness theorem for piecewise constant coefficients with a restriction is stated. We believe that this uniqueness can be generalized up to coefficients of bounded variation. For this aim, we constructed two hyperbolic systems from the time domain approach of the LLSP. One is related to the time domain problem for the ZSSP, and it has a complicated characteristic boundary condition. Thus it is not easy to extend to discontinuous coefficients. However, the other problem has relatively simple boundary condition. In Section 5.4, we discuss this time domain problem in more detail, and give a numerical example to support our conjecture.

5.2 Piecewise constant coefficients

There are infinitely many coefficients in the LLSP which produce the same scattering data if discontinuity for coefficients is allowed. One can obtain these examples from piecewise constant coefficients.

First, we investigate the direct problem for one of the simplest case, one layer coefficient. Recall that the structure of Q is

$$\begin{pmatrix} p & q^* \\ q & -p \end{pmatrix}, \quad p = q_3, \quad q = q_1 + q_2 i, \quad p^2 + |q|^2 = 1.$$

Suppose that Q is a constant on $[0, X]$ and it is Λ elsewhere. From the standard argument of the ODE theory, the solution to

$$\psi_x(x,\zeta) = i\zeta Q(x)\psi(x,\zeta)$$

with boundary condition $\psi(X) = \psi_X$ is given by

$$\psi(x,\zeta) = M(x,\zeta)\psi_X,$$

where,

$$\begin{aligned} M(x,\zeta) &= \exp(i\zeta Q(x-X)) \\ &= \begin{pmatrix} \cos\zeta(X-x) - ip\sin\zeta(X-x) & -iq^*\sin\zeta(X-x) \\ -iq\sin\zeta(X-x) & \cos\zeta(X-x) + ip\sin\zeta(X-x) \end{pmatrix}. \end{aligned}$$

Now, we construct the fundamental matrix for a piecewise constant coefficient . Let $\{x_{-1} = -\infty, x_0 = 0, x_1, \cdots, x_N, x_{N+1} = \infty\}$ be a partition for the real line. Suppose that Q is of the form $Q = \sum_{n=0}^{N+1} Q_k$, where

$$Q_k = \begin{pmatrix} p_k & q_k^* \\ q_k & -p_k \end{pmatrix}, \quad p_k^2 + |q_k|^2 = 1, \quad \text{on } l_k = (x_{k-1}, x_k). \tag{5.2}$$

Denote $h_k = |l_k|$ which is the length of the interval l_k. From the asymptotic behavior of Q, $Q = \Lambda$ on l_0 and l_{N+1}. We can remove this conditions if steplike coefficients are allowed. We deal with, however, standard coefficients in this chapter.

Define

$$\begin{aligned} M_k(\zeta) &= \exp(-i\zeta Q h_k) \\ &= \begin{pmatrix} \cos\zeta(h_k) - ip_k \sin\zeta(h_k) & -iq_k^* \sin\zeta(h_k) \\ -iq_k \sin\zeta(h_k) & \cos\zeta(h_k) + ip_k \sin\zeta(h_k) \end{pmatrix}, \end{aligned}$$

and $M(\zeta)$ by

$$M(\zeta) = M_1(\zeta)M_2(\zeta)\cdots M_n(\zeta) =: \prod_{k=1}^{n} M_k(\zeta).$$

Then it is not difficult to show that the transition matrix $\mathcal{T}$ is given by a linear transformation of the $M(\zeta)$. Indeed, from the definition of the transition matrix (1.7), we have

$$J_+(x,\zeta)\exp(-i\zeta\Lambda X) = J_-(x,\zeta)\mathcal{T}(\zeta)\exp(-i\zeta\Lambda X).$$

The left hand side is the identity matrix at $x = X$. On the other hand the right hand side is $\mathcal{T}(\zeta)\exp(-i\zeta\Lambda X)$ at $x = 0$. The definition of $M(\zeta)$ gives

$$M(\zeta) = \mathcal{T}(\zeta)\exp(-i\zeta\Lambda X). \tag{5.3}$$

This shows that $M(\zeta)$ and $\mathcal{T}(\zeta)$ are equivalent, and the left scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^{N}$ can be represented in terms of $M(\zeta)$. From (5.3), the components of $\mathcal{T}$ are given by

$$a(\zeta) = e^{i\zeta X}M_{11}(\zeta), \quad b(\zeta) = e^{i\zeta X}M_{21}(\zeta),$$

where, M_{ij} is the ij component of M. Since $e^{i\zeta X}$ never vanishes, $a(\zeta)$ and $M_{11}(\zeta)$ have the same zeros, and

$$L(\zeta) = \frac{M_{21}(\zeta)}{M_{11}(\zeta)}, \quad C_{r,n} = \frac{b(\zeta_n)}{\dot{a}(\zeta_n)} = \frac{M_{12}(\zeta_n)}{\dot{M}_{11}(\zeta_n)}.$$

Recall that $\dot{}$ denotes a derivative with respect to given variable.

Now consider the following coefficients of the LLSP. For constants Q_1 and Q_2, define coefficients $Q(x), P(x)$ as follows.

$$Q(x) = Q_1\chi_{[0,h)} + \Lambda\chi_{[0,h)^C}, \tag{5.4}$$

$$P(x) = Q_1\chi_{[0,h)} + Q_2\chi_{[h,2h)} - Q_2\chi_{[2h,3h)} + \Lambda\chi_{[0,3h)^C}, \tag{5.5}$$

for some $h > 0$. Then, obviously Q, P are different coefficients. However, the corresponding matrices M_Q and M_P, thus the left scattering data coincide. Indeed,

$$
\begin{aligned}
M_Q(\zeta) &= \exp(-i\zeta Q_0 h), \\
M_P(\zeta) &= \exp(-i\zeta Q_0 h) \exp(-i\zeta Q_1 h) \exp(i\zeta Q_1 h) \\
&= \exp(-i\zeta Q_0 h).
\end{aligned}
$$

We have the last equality since $-i\zeta Q_1 h$ and $i\zeta Q_1 h$ commute.

We remark that this example for the non-uniqueness does not conflict with the uniqueness for $Q \in \mathcal{Q}^p_{r,n}$, since a piecewise constant coefficient cannot be approximated by $Q_n \in \mathcal{Q}^p_{r,n}$. This example also shows that the one to one correspondence between transition matrices and the scattering data is not valid any more. Obviously, scattering data can be uniquely determined from transition matrices, and conversely transition matrices might be computed from the scattering data due to the spectral representation (1.21) in general. In above example, however,

$$
\mathcal{T}_Q = \exp(-i\zeta Q_0 h) \exp(-i\zeta \Lambda h), \quad \mathcal{T}_P = \exp(-i\zeta Q_0 h) \exp(-i\zeta \Lambda 3h).
$$

Note that this is not a contraction to the spectral representation, since $a(\zeta)$ does not converges to 1 in $\mathbb{C}^+$ if Q is discontinuous. For instance, consider

$$
Q(x) = Q_1 \chi_{[1,2)} + \Lambda \chi_{[1,2)^C},
$$

for $Q_1 = \begin{pmatrix} q_3 & q_1 - q_2 i \\ q_1 + q_2 i & -q_3 \end{pmatrix}$. Then it is easy to obtain that

$$
a(\zeta) = \frac{e^{-2i\zeta}(1 - q_3) + e^{-4i\zeta}(1 + q_3)}{2}.
$$

Obviously, $a(\zeta)$ is an entire function but not converge to 1 as $|\zeta| \to \infty$ in the upper half plane or even on the real line. Thus $L(\zeta)$ is not square integrable function on the real line. Indeed,

$$
\begin{aligned}
L(\zeta) &= \frac{-e^{-i\zeta}(q_1 i - q_2) \sin\zeta}{e^{-3i\zeta}(\cos\zeta - i q_3 \sin\zeta)} \\
&= \frac{q_1 + q_2 i}{q_3 + 1} e^{2i\zeta}(1 - e^{2i\zeta}) \frac{1}{1 - \frac{q_3 - 1}{q_3 + 1} e^{2i\zeta}}.
\end{aligned}
$$

If $q_3 > 0$ is assumed, then $|\frac{q_3-1}{q_3+1}| < 1$ and

$$L(\zeta) = \frac{q_1+q_2 i}{q_3+1}e^{2i\zeta} + \frac{2q_3(q_1+q_2 i)}{(q_3+1)^2}\sum_{k=2}^{\infty}(\frac{q_3-1}{q_3+1})^{k-2}e^{2ik\zeta}.$$

Thus

$$\widehat{L}(t) = \frac{q_1+q_2 i}{q_3+1}\delta(t-2) + \frac{2q_3(q_1+q_2 i)}{(q_3+1)^2}\sum_{k=2}^{\infty}(\frac{q_3-1}{q_3+1})^{k-2}\delta(t-2k).$$

That is, $\widehat{L}$ is a distribution and its support is $\{2k\}_{k=1}^{\infty}$.

In general, we believe the following statement.

Conjecture 5.2.1. *Suppose that $Q(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ in the LLSP. Then for the left and right reflection coefficient $L(\zeta), R(\zeta)$,*

$$\widehat{L}, \widehat{R} \in H^{-1}(\mathbb{R}),$$

where, $H^{-1}(\mathbb{R})$ is the dual space of $H^1(\mathbb{R})$.

The next lemma may give a clue when the uniqueness fails for piecewise constant coefficients .

Lemma 5.2.2. *Suppose that*

$$Q(x) = \sum_{k=0}^{N+1} Q_k \chi_{l_k}$$

and $Q_k \neq -Q_{k+1}$ for all $k = 0, 1, \cdots, N$. Then $M(\zeta)$ that is defined in (5.3) uniquely determines $Q(x)$. Here Q_k and l_k are defined in (5.2) and $Q_0 = Q_{N+1} = \Lambda$.

Proof. We prove this lemma by the mathematical induction. First we state the following claim which is used several times in other steps.

Claim (1)**.** *Two constant coefficients Q and P in the LLSP commute with each other if and only if $Q = \pm P$.*

The necessary condition is trivial. For the sufficient condition, let us consider a unitary matrix U such that $QU = U\Lambda_1$ and $PU = U\Lambda_2$. The commutativity of Q and P guarantees the existence of U([32]) . Here Λ_i is Λ or $-\Lambda$. The multiplication of $U^\dagger$ to the left and U to the right of each side of $QP = PQ$ gives,

$$\Lambda_1\Lambda_2 = U^\dagger PQU.$$

Thus we have $PQ = \pm I$, or

$$P = \pm Q.$$

Claim (2)**.** *Suppose that* $\exp(-i\zeta Q_1 h_1) = \exp(-i\zeta P_1 m_1)\exp(-i\zeta P_2 m_2)$ *for* $P_1 \neq -P_2$ *and* $h_1, m_k > 0$*. Then* $P_1 = P_2 = Q_1$ *and* $h_1 = m_1 + m_2$*.*

By comparing the coefficients of ζ and ζ^2 of expansions of each sides, we have

$$Q_1 h_1 = P_1 m_1 + P_2 m_2, \tag{5.6}$$

$$\frac{(Q_1 h_1)^2}{2!} = \frac{(P_1 m_1)^2}{2!} + \frac{(P_2 m_2)^2}{2!} + P_1 m_1 P_2 m_2. \tag{5.7}$$

Thus, $P_1 P_2 = P_2 P_1$. This implies $P_1 = \pm P_2$. But we assume that $P_1 \neq -P_2$. Together with (5.6), $P_1 = P_2$ and $h_1 = m_1 + m_2$.

Claim (3)**.** *Suppose that for any* $n \leq N$, $\exp(-i\zeta Q_1 h_1) = \prod_{k=1}^{n} \exp(-i\zeta P_k m_k)$ *implies* $Q_1 = P_k$ *for all* $1 \leq k \leq n$ *and* $h_1 = \sum_{k=1}^{n} m_k$ *provided* $P_k \neq -P_{k+1}$ *for all* k *and* $h_1, m_k > 0$*. Then this is also true for* $n = N+1$*. Here the matrix product* $\prod_{k=1}^{N} A_k$ *is defined by* $A_1 A_2 \cdots A_N$*.*

Suppose that

$$\exp(-i\zeta Q_1 h_1) = \prod_{k=1}^{N+1} \exp(-i\zeta P_k m_k). \tag{5.8}$$

Let $A = \sum_{k=2}^{N+1} P_k m_k$. Since $\operatorname{trace}(A) = 0$ and $\det(A) \leq 0$, the eigenvalues of A are $\pm\lambda$ for some $\lambda \in \mathbb{R}$. Suppose that $\lambda = 0$. Then $A = 0$, and

$$\exp(-i\zeta(-P_2)m_2) = \prod_{k=3}^{N+1} \exp(-i\zeta P_k m_k).$$

From the induction hypothesis, $-P_2 = P_k$ for all $3 \leq k \leq N+1$, which is a contraction to $-P_2 \neq P_3$. Thus $\lambda \neq 0$. Since A is a Hermitian matrix, for a unitary matrix U

$$U^\dagger A U = \lambda\Lambda,$$

or

$$A^2 = \lambda^2 I.$$

By the similar way to obtain (5.6), (5.7)

$$\begin{aligned} Q_1 h_1 &= P_1 m_1 + A, & (5.9) \\ \frac{(Q_1 h_1)^2}{2!} &= \frac{(P_1 m_1)^2}{2!} + \frac{A^2}{2!} + P_1 m_1 A. & (5.10) \end{aligned}$$

Comparing the square of (5.9) and (5.10) yields

$$P_1 A = A P_1.$$

This together with (5.9) implies that P_1 and Q_1 commute with each other. Thus $P_1 = \pm Q_1$. If $P_1 = -Q_1$, then (5.8) can be rewritten as

$$\exp(-i\zeta Q_1(h_1 + m_1)) = \prod_{k=2}^{N+1} \exp(-i\zeta P_k m_k).$$

By the hypothesis of the claim $Q_1 = P_k$ for $2 \le k \le N+1$, and $P_1 = -P_2$. But this contradicts to the assumption of the claim. Thus P_1 should be equal to Q_1. We rewrite (5.8) as

$$\exp(-i\zeta Q_1(h_1 - m_1)) = \prod_{k=2}^{N+1} \exp(-i\zeta P_k m_k).$$

If $h_1 > m_1$, then we prove the claim by the induction hypothesis. If $h_1 < m_1$, then similarly to the case of $P_1 = -Q_1$, we have $P_1 = -P_2$ which gives a contraction also. Consider the remain case, $h_1 = m_1$. In this case, we have

$$\exp(-i\zeta(-P_2)m_2) = \prod_{k=3}^{N+1} \exp(-i\zeta P_k m_k).$$

Again, the hypothesis implies that $-P_2 = P_3$ which contradicts to the assumption.

Claim (4)**.** *Suppose that for any N, M, $\prod_{k=1}^{N} \exp(-i\zeta Q_k h_k) = \prod_{k=1}^{M} \exp(-i\zeta P_k m_k)$ and any adjacent Q_k and P_k do not commute and $h_k, m_k > 0$. Then $N = M$, $Q_k = P_k$ and $h_k = m_k$ for all $1 \le k \le N$.*

We prove this statement by the mathematical induction again. Claim 3 shows that this holds for $N = 1$. Now assume that for given N, the above statement is valid for any $n \le N$

and any arbitrary M, and

$$\prod_{k=1}^{N+1} \exp(-i\zeta Q_k h_k) = \prod_{k=1}^{M} \exp(-i\zeta P_k m_k).$$

Then,

$$\prod_{k=1}^{N} \exp(-i\zeta Q_k h_k) = \prod_{k=1}^{M} \exp(-i\zeta P_k m_k) \exp(-i\zeta(-Q_{N+1})h_{N+1}). \tag{5.11}$$

If $P_M \neq \pm Q_{N+1}$, then from the hypothesis $Q_N = -Q_{N+1}$. If $P_M = -Q_{N+1}$, then

$$\prod_{k=1}^{N} \exp(-i\zeta Q_k h_k) = \prod_{k=1}^{M-1} \exp(-i\zeta P_k m_k) \exp(-i\zeta(-Q_{N+1})(m_M + h_{N+1})),$$

thus $Q_N = -Q_{N+1}$. From the assumption that any adjacent Q_k do not commute, $P_M = Q_{N+1}$. Now assume that $h_{N+1} \neq m_M$, say $h_{N+1} < m_M$. Then from (5.11),

$$\prod_{k=1}^{N} \exp(-i\zeta Q_k h_k) = \prod_{k=1}^{M} \exp(-i\zeta P_k m_k) \exp(-i\zeta(Q_{N+1})(m_M - h_{N+1})),$$

which implies $Q_N = Q_{N+1}$. In case of $h_{N+1} > m_M$, we have $Q_N = -Q_{N+1}$. Hence, $h_{N+1} = m_M$. □

Lemma 5.2.2 states the one to one correspondence of piecewise constant coefficients $Q(x)$ with a certain restriction and the matrices $M(\zeta)$. We might have a similar result for the standard left (right) scattering data $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^{N}$ and $Q(x)$. For this, we need to show that $M(\zeta)$ is uniquely determined from the standard scattering data. This might be based on development of a new spectral representation such as (1.26), and the knowledge of asymptotic behaviors of $L(\zeta)$ or $a(\zeta)$ for $\zeta \in \mathbb{C}^+$.

Conjecture 5.2.3. *Suppose that Q in the LLSP is a piecewise constant, and any adjacent layers, say Q_k and Q_{k+1}, do not commute with each other, i.e.*

$$Q_k \neq -Q_{k+1}.$$

Then the standard left (right) scattering data, $\{L(\zeta), \zeta_n, C_{r,n}\}_{n=1}^{N}$ uniquely determines Q.

5.3 Two time domain problems

In previous section, we mainly discussed the LLSP with piecewise constant coefficients. Especially, the relation of coefficients and $M(\zeta)$ was discussed. Even though we conjecture that $M(\zeta)$ might be computed from the scattering data, Theorem 5.2.2 is not helpful to restore $Q(x)$ from the scattering data. Thus we have to develop a new method. One of the approaches is transforming the LLSP to a time domain problem. In this section, we formally design two time domain problems. Mathematical details will be given in the next section.

5.3.1 Time domain problem I

For simplicity of current discussion, we assume that Q in the LLSP,

$$\psi_x = i\zeta Q\psi \tag{5.12}$$

involves no bound states, $Q - \Lambda$ is compactly supported, say in $[0, X]$, and smooth enough. Applying the Fourier transform $\dfrac{1}{2\pi}\displaystyle\int \cdot\, e^{-i\zeta t}$ to (5.12) yields

$$u_x(x,t) + Q(x)u_t(x,t) = 0, \tag{5.13}$$

where $u(x,t) = \dfrac{1}{2\pi}\displaystyle\int \psi(x,\zeta)e^{-i\zeta t}d\zeta$. If we define the initial condition as

$$u = \begin{pmatrix} \delta(t-x) \\ 0 \end{pmatrix} \quad \text{for } t < 0, \tag{5.14}$$

then the boundary condition for (5.13) is given by

$$u(0,t) = \begin{pmatrix} 0 \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} L(\zeta)e^{-i\zeta t}d\zeta \end{pmatrix}, \quad \text{for } t > 0. \tag{5.15}$$

Note that since $u^{(1)}(x,t)$ is incoming wave from the left, $u^{(1)}(x,t) = 0$ for $x < 0$ and $t \neq x$. Thus $u^{(1)}(0,t) = \lim_{x\to 0^-} u^{(1)}(x,t) = 0$. We assume that $u^{(1)}(0,t)$ is continuous on $\{t > x\}$.

It is well known that (5.13) has unique solution on $\{(x,t) : x > 0, t > 0, t > x\}$ with the boundary conditions (5.15). We seek the suitable additional boundary condition for (5.13) on

$x = t$ so that it is an over-determined problem. Consider the propagation of singularity ([16]). Let

$$u = \sum_{k=0}^{N} f_k(t-x)u_k(x) + smooth\ terms, \tag{5.16}$$

where $f_0(x) = \delta(x), f_1 = H(x)$ and so on. Then,

$$Qu_0 - u_0 = 0, \tag{5.17}$$

$$Qu_k - u_k = -u_{k-1,x} \quad k \geq 1. \tag{5.18}$$

From the initial condition (5.15), we set

$$u_0(0^+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_1^{(1)}(0^+) = 0. \tag{5.19}$$

Now we investigate the uniqueness of u_0. From the equation (5.17), u_0 should be an eigenvector for Q corresponding to eigenvalue 1. Since eigenvector is not unique, we have to find a side condition. Multiplication $u_0^\dagger$ to the left of each side of (5.18) for $k = 1$ gives

$$u_0^\dagger Q u_1 - u_0^\dagger u_1 = -u_0^\dagger u_{0,x} \tag{5.20}$$

$$0 = -u_0^\dagger u_{0,x}. \tag{5.21}$$

From (5.21) and (5.19),

$$|u_0| = 1,$$

and $u_{0,x}$ is in the orthogonal space of span$\{u_0\}$ for each fixed x.

Suppose that v, w are column vectors satisfying (5.17) and (5.19) with $|v| = |w| = 1$. Then $v = e^{i\theta(x)}w$ for a real valued function $\theta(x)$ such that $\theta(0) = 0$. Indeed, it is easy to see that v, w are linearly dependent due to (5.17), i.e. $v = \vartheta(x)w$ for some $\vartheta(x)$ with $\vartheta(0) = 1$. From the conditions $|v| = |w| = 1$, $\vartheta(x) = e^{i\theta(x)}$ for a real valued function $\theta(x)$ such that $\theta(0) = 0$. Let $u_0 = e^{i\theta}v$. Then $u_{0,x} = i\theta_x e^{i\theta}v + e^{i\theta}v_x$, and

$$\begin{aligned} < u_0, u_{0,x} > &= < e^{i\theta}v, i\theta_x e^{i\theta}v + e^{i\theta}v_x > \\ &= i\theta_x. \end{aligned}$$

Here, we denote by $< \cdot, \cdot >$ the standard inner product in $\mathbb{C}^2$. Since $< u_0, u_{0,x} >= 0$, θ is identically zero. This implies the uniqueness of u_0.

Another method to check the uniqueness and existence is to use the transformation $\mathfrak{F}$ which is discussed in Chapter 3. Let F_1 be the the left column vector of the matrix F in $\mathfrak{F}$. From construction of F and Lemma 3.2.3, we have

$$F_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |F_1| = 1, \quad QF_1 = F_1,$$

and

$$QF_{1,x} = -F_{1,x}. \tag{5.22}$$

Apply $F_1^\dagger$ to (5.22).

$$\begin{aligned} < F_1, QF_{1,x} > &= - < F_1, F_{1,x} >, \\ < F_1, F_{1,x} > &= - < F_1, F_{1,x} >, \\ < F_1, F_{1,x} > &= 0. \end{aligned}$$

Hence

$$F_1 = u_0.$$

We showed that F thus F_1 exists uniquely as long as $Q \in \mathcal{Q}^p_{r,n}$.

Next, we compute u_1. Since $u_{0,x}$ is orthogonal to u_0, it is an eigenvector of Q corresponding to eigenvalue -1. Thus for scalar functions $\eta, \overline{\eta}$

$$u_1 = \eta u_0 + \overline{\eta} u_{0,x}. \tag{5.23}$$

Substitution this representation into (5.18) for $k = 1$ gives,

$$\begin{aligned} Q(\eta u_0 + \overline{\eta} u_{0,x}) - (\eta u_0 + \overline{\eta} u_{0,x}) &= -u_{0,x}, && (5.24) \\ -2\overline{\eta} u_{0,x} &= -u_{0,x}. && (5.25) \end{aligned}$$

We have $\overline{\eta} = \frac{1}{2}$. Now operate $u_0^\dagger$ to (5.18) for $k = 2$.

$$u_0^\dagger Q u_2 - u_0^\dagger u_2 = -u_0^\dagger u_{1,x}.$$

Since $Qu_0 = u_0, Q^\dagger = Q$,

$$< u_0, u_{1,x} >= 0.$$

With representation of (5.23),

$$\begin{aligned} < u_0, \eta_x u_0 + \eta u_{0,x} + \frac{1}{2} u_{0,xx} > &= 0, \\ \eta_x + \frac{1}{2} < u_0, u_{0,xx} > &= 0. \end{aligned}$$

the initial condition (5.19) gives $\eta(0) = 0$, and

$$\begin{aligned} \eta(x) &= -\frac{1}{2} \int_0^x < u_0, u_{0,xx} >, \\ &= -\frac{1}{2} < u_0, u_{0,x} > |_0^x + \frac{1}{2} \int_0^x < u_{0,x}, u_{0,x} >, \\ &= \frac{1}{2} \int_0^x < u_{0,x}, u_{0,x} > . \end{aligned}$$

Thus,

$$u_1(x) = \frac{1}{2} u_0 \int_0^x < u_{0,x}, u_{0,x} > + \frac{1}{2} u_{0,x}. \tag{5.26}$$

From the representation of (5.16), the additional boundary condition is given by

$$u(x, x^+) = u_1(x).$$

We summarize above argument as follows.

Time domain problem I for the LLSP.

$$u_x(x,t) + Q(x) u_t(x,t) = 0, \qquad 0 < x < t, \tag{5.27}$$

$$u(0,t) = \begin{pmatrix} 0 \\ \frac{1}{2\pi} \int_{\mathbb{R}} L(\zeta) e^{-i\zeta t} d\zeta \end{pmatrix}, \qquad t > 0, \tag{5.28}$$

$$u(x,x) = u_1(x), \qquad x > 0. \tag{5.29}$$

The knowledge of the relation of $u_1(x)$ and $Q(x)$ is essential to solve the inverse problem. Serval methods might be applied. One of them is following.

1. *From* (5.26),

$$\begin{aligned} < u_1, u_1 > &= < \frac{1}{2} u_0 \int_0^x < u_{0,x}, u_{0,x} > + \frac{1}{2} u_{0,x}, \ \frac{1}{2} u_0 \int_0^x < u_{0,x}, u_{0,x} > + \frac{1}{2} u_{0,x} > \\ &= \frac{1}{4} (\int_0^x < u_{0,x}, u_{0,x} >)^2 + \frac{1}{4} < u_{0,x}, u_{0,x} > . \end{aligned}$$

Solve this nonlinear differential equation for $\int_0^x < u_{0,x}, u_{0,x} >$.

2. *Compute* u_0 *from* (5.26).

3. *Recover* Q *by* $G\Lambda G^{-1}$, *where* $G = [u_0 \; u_{0,x}]$.

We remark that Time domain problem I is equivalent the time domain problem for the ZSSP if the coefficient Q is in the class $\mathcal{Q}^p_{r,n}$. Recall the time domain problem for the ZSSP.

$$v_x(x,t) + \Lambda v_t(x,t) = \Lambda S v, \qquad 0 < x < t, \tag{5.30}$$

$$v(0,t) = \begin{pmatrix} 0 \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} L(\zeta)e^{-i\zeta t}d\zeta \end{pmatrix}, \qquad t > 0, \tag{5.31}$$

$$v^{(2)}(x,x) = \frac{1}{2}s(x), \qquad x > 0. \tag{5.32}$$

It is not difficult to check that (5.27) is transformed to (5.30) by defining $v = F^\dagger u$, where F is the matrix corresponding transformation $\mathfrak{F}$. Also we can obtain the additional boundary condition (5.32) for the ZSSP from $(F^\dagger u_1)^{(2)}$. Indeed,

$$\begin{aligned} (F^\dagger u_1)^{(2)} &= \frac{1}{2}(F^\dagger u_0 \int_0^x < u_{0,x}, u_{0,x} > + F^\dagger u_{0,x})^{(2)} \\ &= \frac{1}{2}F_2^\dagger u_{0,x} \\ &= \frac{1}{2}F_2^\dagger F_{1,x}. \end{aligned}$$

From the transformation $\mathfrak{F}$ i.e. $\Lambda S = F_x^\dagger F$,

$$(F^\dagger u_1)^{(2)} = \frac{1}{2}s. \tag{5.33}$$

5.3.2 Time domain problem II

In Time domain problem I, it is not easy to find a direct relation of Q and u_1. Furthermore, the boundary condition on $x = 0$ may be a distribution if Q has a jump as mentioned in Section 5.2. Thus, we have to construct a new equation which is equivalent to (5.27)-(5.29) and can be generalized for discontinuous coefficients. A new problem may be constructed by a simple change of variable.

Let

$$w(x,t) = \int_x^t u(x,s)ds + \alpha(x).$$

Here we assume $\alpha(x)$ is a smooth enough for a moment, which is defined later. Then

$$\begin{aligned} w_t(x,t) &= u(x,t), \\ w_x(x,t) &= \int_x^t u_x(x,s)ds - u(x,x) + \alpha_x(x), \end{aligned}$$

and

$$\begin{aligned} Q(x)w_x(x,t) &= \int_x^t Q(x)u_x(x,s)ds - Qu(x,x) + Q\alpha_x(x), \\ &= -\int_x^t u_t(x,s)ds - Qu(x,x) + Q\alpha_x(x), \\ &= -u(x,t) + u(x,x) - Qu(x,x) + Q\alpha_x(x), \\ &= -w_t(x,t) + u(x,x) - Qu(x,x) + Q\alpha_x(x). \end{aligned}$$

Define $\alpha(x)$ as

$$\alpha_x(x) = -Q(x)u(x,x) + u(x,x), \quad \alpha(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.34}$$

Then (5.27)-(5.29) are changed to

$$w_x(x,t) + Q(x)w_t(x,t) = 0, \qquad 0 < x < t, \tag{5.35}$$

$$w(0,t) = \begin{pmatrix} 1 \\ \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} L(\zeta)e^{-i\zeta s}d\zeta ds \end{pmatrix}, \qquad t > 0, \tag{5.36}$$

$$w(x,x) = \alpha(x), \qquad x > 0. \tag{5.37}$$

Here, we use $Q^2 = I$ to have (5.35).

As we did in Time domain problem I, the relation of $\alpha(x)$ and the coefficient $Q(x)$ should be verified to solve the inverse problem. The equation for α, (5.34), and the equation for u_1, (5.18) for $k = 1$ together with (5.19) and (5.29) show that

$$\alpha(x) = u_0(x).$$

Although u_0, thus α, is the same as the first column vector of F, we can find relation of Q and α without considering the transformation $\mathfrak{F}$. This is important when we consider discontinuous coefficients because the map $\mathfrak{F}$ is not valid for discontinuous coefficients as we pointed out in Section 5.1.

Since $\begin{pmatrix} 1+q_3 \\ q_1+q_2 i \end{pmatrix}$ is an eigenvector of Q corresponding to the eigenvalue 1, there exists a scalar function ϱ such that

$$\alpha = \begin{pmatrix} \alpha^{(1)} \\ \alpha^{(2)} \end{pmatrix} = \varrho \begin{pmatrix} 1+q_3 \\ q_1+q_2 \end{pmatrix}.$$

We assume that for any x,

$$q_3(x) \neq -1. \tag{5.38}$$

Then

$$\frac{\alpha^{(2)}}{\alpha^{(1)}} = \frac{q_1+q_2 i}{1+q_3}.$$

Hence, we design another time domain problem from Time domain problem I.

Time domain problem II for the LLSP.

Suppose that q_3+1 never vanishes. Then

$$w_x(x,t) + Q(x)w_t(x,t) = 0, \qquad 0 < x < t, \tag{5.39}$$

$$w(0,t) = \begin{pmatrix} 1 \\ \int_0^t \widehat{L}(s)ds \end{pmatrix}, \qquad t > 0, \tag{5.40}$$

$$w(x,x) = \alpha(x), \qquad x > 0, \tag{5.41}$$

and Q and α related by

$$\frac{\alpha^{(2)}}{\alpha^{(1)}} = \frac{q_1+q_2 i}{1+q_3}. \tag{5.42}$$

We seek Q from given data $L(\zeta)$ or $\widehat{L}(s) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} L(\zeta)e^{-i\zeta s}d\zeta$.

We remark that Time domain problem II might be derived from direct argument of the propagation of singularity. From the definition of w, we may assume that

$$w(x,t) = w_1(x)H(t-x) + w_2(x)|t-x|^{+} + \textit{smooth terms}$$

instead of (5.16). Here, w_k is the column vector and $H(x)$ is the Heaviside function and $|x|^{+} = x\chi_{\{x>0\}}$.

5.4 Uniqueness for Time domain problem II

We devote this section to discussing the following conjecture about a general uniqueness for the LLSP with discontinuous coefficient based on Time domain problem II. Also a numerical example is given at the end of this section.

Conjecture 5.4.1. *Suppose that $L(\zeta)$ is the left reflection coefficient for the LLSP with coefficient Q. If*

(1) $Q(x) - \Lambda$ is compactly supported, say in $[0, X]$,

(2) there are no bound states,

(3) $|a(\zeta)| > 0$ on the real line,

(4) $\int_0^t \widehat{L}(s)ds$ is of bounded variation in $[0, 2X]$,

(5) $\sup_t |\int_{t^-}^{t^+} \widehat{L}(s)ds| < M_1$ for some M_1.

Then, $L(\zeta)$ uniquely determines a coefficient $Q(x)$ such that

(1) $Q(x)$ is discontinuous only at the points at which $\int_0^t \widehat{L}(s)ds$ is discontinuous,

(2) $\sup_x |Q(x^+) - Q(x^-)| < M_2$ for some $M_2 < \sqrt{2}$.

First, we show that direct Time domain problem II is well defined for a weaker regularity than assumed regularity when it is constructed. From this, we define a map $\mathfrak{L}$ as follows.

$$\mathfrak{L} : \begin{pmatrix} q_1 + q_2 i \\ q_3 \end{pmatrix} \mapsto \begin{pmatrix} \dfrac{2u_1^* u_2}{|\mathbf{u}|^2} \\ \dfrac{|u_1|^2 - |u_2|^2}{|\mathbf{u}|^2} \end{pmatrix},$$

where $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the trace of $w(x,t)$, a solution of (5.39)-(5.40) with Q whose entries are given by q_1, q_2 and q_3 in L^2, on the line segment $t = x$ for $0 < x < X$. Suppose that $\mathfrak{L}$ has a fixed point. Then

$$\begin{aligned} \frac{q_1 + q_2 i}{1 + q_3} &= [\frac{2u_1^* u_2}{|\mathbf{u}|^2}]/[1 + \frac{|u_1|^2 - |u_2|^2}{|\mathbf{u}|^2}] \\ &= \frac{2u_1^* u_2}{2|u_1|^2} = \frac{u_2}{u_1}. \end{aligned}$$

Thus, the characteristic boundary condition (5.42) holds. Conversely, from (5.42)

$$|q_1|^2 + |q_2|^2 = (1 + q_3)^2 |\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2.$$

Since $|q_1|^2 + |q_2|^2 + |q_3|^2 = 1$,

$$(|\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2 + 1)q_3^2 + 2|\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2 q_3 + |\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2 - 1 = 0.$$

So we have

$$q_3 = [1 - |\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2]/[1 + |\frac{\alpha^{(2)}}{\alpha^{(1)}}|^2] = \frac{|\alpha^{(1)}|^2 - |\alpha^{(2)}|^2}{|\alpha|^2},$$

assuming (5.38). This representation, together with (5.42) gives

$$q_1 + q_2 i = \frac{2|\alpha^{(1)}|^2}{|\alpha|^2}\frac{\alpha^{(2)}}{\alpha^{(1)}} = \frac{2{\alpha^{(1)}}^*\alpha^{(2)}}{|\alpha|^2}.$$

Thus, if there is $w(x, x)$ satisfying (5.39)-(5.42) for some Q, it should be a fixed point of the map $\mathfrak{L}$.

Our aim is to show that $\mathfrak{L}$ is a contraction mapping on a certain complete space $\mathcal{X}$. Then it has a unique fixed point, which gives the uniqueness and existence of Time domain problem II. Together with Conjecture 5.2.1, then, we might show the uniqueness of the LLSP in frequency domain for piecewise smooth coefficients. Note that Conjecture 5.4.1 does not contradict to the example shown in Section 5.2, since the maximum value of jumps is $\sqrt{2}$ for the coefficient P defined in (5.5).

5.4.1 Direct problem

Consider the direct problem of Time domain problem II.

$$w_x + Q(x)w_t = 0 \quad \text{on } \mathbb{T}, \qquad w(0, t) = w_0(t) \ (0 < t < 2X). \tag{5.43}$$

Here, $\mathbb{T}$ denotes the domain bounded by $x = t, x = 0$, and $t = 2X - x$ (see Figure 5.1). If $Q(x)$ and $w_0(t)$ are smooth enough, then one may show the existence and uniqueness of the classical solution by standard energy estimates for hyperbolic system, see e.g. [34]. For a less regularity of $Q(x)$ and $w_0(t)$, however, w may not be differentiable. We need to define a weak solution.

Suppose that w is a classical solution with smooth Q and w_0. Then for a test function $\eta(x,t) \in \mathcal{C}$

$$\int_{\mathbb{T}} (w_x + Qw_t)^\dagger \eta = 0,$$

where,

$$\mathcal{C} = \{\eta \in C^\infty(\mathbb{T}) : \eta(x,\cdot) \in C_c^\infty(x, 2X - x),\ 0 \le x < X\}.$$

Integration by parts yields

$$\int_{\mathbb{T}} w^\dagger(\eta_x + Q\eta_t) + \int_0^{2X} w_0^\dagger \eta|_{x=0} dt = 0. \tag{5.44}$$

This equation makes sense even if $w \in L^2(\mathbb{T}), Q \in L^2(0,X)$ and $w_0 \in L^2(0,2X)$. So we can define a weak solution of (5.43) as (5.44).

Definition 5.4.2. *$w \in L^2(\mathbb{T})$ is a weak solution to*

$$w_x + Q(x)w_t = 0 \quad \textit{on } \mathbb{T}, \qquad w(0,t) = w_0(t)\ (0 < t < 2X),$$

provided for any $\eta \in \mathcal{C}$,

$$\int_{\mathbb{T}} w^\dagger(\eta_x + Q\eta_t) + \int_0^{2X} w_0^\dagger \eta|_{x=0} dt = 0.$$

The main theorem in this section is the existence and uniqueness for the direct problem as follows.

Proposition 5.4.3. *Suppose that $Q \in L^2(0,X)$ and $w_0 \in L^2(0,2X)$. Then there uniquely exists a weak solution $w \in L^2(\mathbb{T})$ for (5.43).*

We prove the existence and uniqueness separately after Lemma 5.4.5 and Lemma 5.4.6.

Lemma 5.4.4. *Suppose that $Q = \sum_{k=1}^{N+1} Q_k \chi_{[x_{k-1},x_k)}$ for a constant Q_k, $x_0 = 0$ and $x_{N+1} = X$. Then for $w_0(t) \in H^1(0,2X)$ there exists a weak solution w to (5.43) such that*

$$||w||_{H^1(\mathbb{T})} \le \sqrt{2X}||w_0||_{H^1(0,2X)}. \tag{5.45}$$

Since the domain is bounded, w and w_0 are in $W^{1,1}(\mathbb{T})$ and $W^{1,1}(0,2X)$ respectively. In particular,

$$||w(x,\cdot)||_{L^1(x,2X-x)} \le ||w_0||_{L^1(0,2X)}, \tag{5.46}$$

and

$$||w||_{L^1(\mathbb{T})} \leq X||w_0||_{L^1(0,2X)}. \tag{5.47}$$

Proof. For each $Q_k = \begin{pmatrix} p_k & q_k^* \\ q_k & -p_k \end{pmatrix}$, define a Hermitian matrix[1]

$$F_k = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+p_k} & -\dfrac{q_k^*}{\sqrt{1+p_k}} \\ \dfrac{q_k}{\sqrt{1+p_k}} & \sqrt{1+p_k} \end{pmatrix},$$

so that

$$Q_k F_k = F_k \Lambda.$$

It is obvious that a solution to

$$v_x + \Lambda v_t = 0, \quad v(x_{k-1}, t) = v_0(t)$$

on $\mathbb{T}_k = \{(x,t) \in \mathbb{T} : x_{k-1} < x < x_k\}$ is given by

$$v(x,t) = \begin{pmatrix} v_0^{(1)}(t - x + x_{k-1}) \\ v_0^{(2)}(t + x - x_{k-1}) \end{pmatrix}.$$

Define a translation operator $\mathcal{P}_k$,

$$\mathcal{P}_k \begin{pmatrix} f(\cdot) \\ g(\cdot) \end{pmatrix} = \begin{pmatrix} f(t - x + x_{k-1}) \\ g(t + x - x_{k-1}) \end{pmatrix}.$$

Then $\mathcal{P}$ satisfies the following properties.

For two-component vector functions $u(\cdot), v(\cdot) \in H^1(x_{k-1}, 2X - x_{k-1})$,

$$(\mathcal{P}_k u)_t = \mathcal{P}_k(\dot{u}), \tag{5.48}$$

$$(\mathcal{P}_k u)_x = -\Lambda \mathcal{P}_k(\dot{u}), \tag{5.49}$$

$$\mathcal{P}_k(\alpha u \pm \beta v) = \alpha \mathcal{P}_k u \pm \beta \mathcal{P}_k v \quad \text{for } \alpha, \beta \in \mathbb{C}. \tag{5.50}$$

Recall that $\dot{}$ denotes the derivative with respect to given variable.

[1] We assume that $q_3 \neq -1$, see (5.38).

With this notation, define $w(x,t)$ on $\mathbb{T}$ by

$$w(x,t) = F_k \mathcal{P}_k(F_k^\dagger w_{k-1}) \quad \text{on } \mathbb{T}_k,$$

where $w_{k-1}(\cdot) = w(x_{k-1}^-, \cdot)$ for $2 \le k \le N+1$. It is not difficult to show that $w(x,t)$ satisfies (5.44). Note that $w_0 \in H^1$ and $w(x_k^-, t) = w(x_k, t)$ imply $w(x,t) \in H^1(\mathbb{T}_k)$. Thus $w \in H^1(\mathbb{T})$ since w is continuous across $x = x_k$. Moreover, $w(x,x) \in H^1(0,X)$.

Let $u(\cdot)$ be a two-component vector function in $H^1(x_{k-1}, 2X - x_{k-1})$.

$$\begin{aligned}
\int_{\mathbb{T}_k} |\mathcal{P}_k(u)|^2 &= \int_{x_{k-1}}^{x_k} \int_x^{2X-x} |u^{(1)}(t - x + x_{k-1})|^2 + |u^{(2)}(t + x - x_{k-1})|^2 dtdx, \\
&\le \int_0^{x_k - x_{k-1}} \int_{x_{k-1}}^{2X - x_{k-1}} |u^{(1)}(t-y)|^2 + |u^{(2)}(t+y)|^2 dtdy, \\
&\le (x_k - x_{k-1})||u||_{L^2}^2.
\end{aligned}$$

Since $||-\Lambda u||_{L^2} = ||u||_{L^2}$, from (5.48) and (5.49)

$$||\mathcal{P}_k(u)||_{H^1(\mathbb{T}_k)}^2 \le 2(x_k - x_{k-1})||u||_{H^1}^2.$$

Since F_k is a Hermitian matrix, $||F_k u||_{H^1} = ||F_k^* u||_{H^1} = ||u||_{H^1}$. Thus

$$||w||_{H^1(\mathbb{T}_k)}^2 \le 2(x_k - x_{k-1})||w_{k-1}||_{H^1(x_{k-1}, 2X - x_{k-1})}^2. \tag{5.51}$$

Now we estimate $||\dot{w}_{k-1}||_{L^2(x_{k-1}, 2X - x_{i-1})}$.

$$\begin{aligned}
||\dot{w}_{k-1}||_{L^2(x_{k-1}, 2X-x_{k-1})}^2 &= \int_{x_{k-1}}^{2X - x_{k-1}} |\dot{w}_{k-1}(s)|^2 ds \\
&= \int_{x_{k-1}}^{2X - x_{k-1}} |w_t(x_{k-1}, s)|^2 ds \\
&= \int_{x_{k-1}}^{2X - x_{k-1}} |F_{k-1}\mathcal{P}_{k-1}(F_{k-1}^\dagger \dot{w}_{k-2})(x_{k-1}, s)|^2 ds \\
&= \int_{x_{k-1}}^{2X - x_{k-1}} |(F_{k-1}^\dagger \dot{w}_{k-2})^{(1)}(s - x_{k-1} + x_{k-2})|^2 \\
&\qquad + |(F_{k-1}^\dagger \dot{w}_{k-2})^{(2)}(s + x_{k-1} - x_{k-2})|^2 ds \\
&\le \int_{x_{k-2}}^{2X - x_{k-2}} |(F_{k-1}^\dagger \dot{w}_{k-2})^{(1)}(t)|^2 + |(F_{k-1}^\dagger \dot{w}_{k-2})^{(2)}(t)|^2 dt \\
&= ||F_{k-1}^\dagger \dot{w}_{k-2}||_{L^2(x_{k-2}, 2X - x_{k-2})}^2 = ||\dot{w}_{k-2}||_{L^2(x_{k-2}, 2X - x_{k-2})}^2.
\end{aligned}$$

Similarly,

$$||w_{k-1}||^2_{L^2(x_{k-1},2X-x_{k-1})} \leq ||w_{k-2}||^2_{L^2(x_{k-2},2X-x_{k-2})}.$$

By induction,

$$||w_{k-1}||^2_{H^1(x_{k-1},2X-x_{k-1})} \leq ||w_0||^2_{H^1(0,2X)}.$$

This inequality together with (5.51) gives (5.45). Similarly, we can obtain (5.46) and (5.47). $\square$

Lemma 5.4.5. *Suppose $Q(x) \in L^2(0,X)$, and $w_0(t) \in H^1(0,2X)$. Then a weak solution $w(x,t)$ of (5.43) exists in $H^1(\mathbb{T})$ satisfying*

$$||w(x,\cdot)||_{L^1(x,2X-x)} \leq ||w_0||_{L^1(0,2X)}, \tag{5.52}$$

and

$$||w||_{L^1(\mathbb{T})} \leq X||w_0||_{L^1(0,2X)}. \tag{5.53}$$

Proof. For given $Q \in L^2$, there exists a sequence of piecewise constant coefficient $\{Q_n\}$ converging to Q in L^2. From Lemma 5.4.4, there is $w_n \in H^1(\mathbb{T})$ solving (5.43) for each Q_n, and

$$||w_n||_{H^1(\mathbb{T})} \leq \sqrt{2X}||w_0||_{H^1(0,2X)}.$$

Thus we can find a subsequence w_{n_k} such that

$$w_{n_k} \rightharpoonup w \quad \text{in } H^1(\mathbb{T}).$$

We claim that w is a weak solution to (5.43).

For any $\eta \in \mathcal{C}$,

$$\begin{aligned} & < (w_x + Qw_t) - (w_{n_k,x} + Q_{n_k} w_{n_k,t}), \eta >_{L^2} \\ = \; & < w_x - w_{n_k,x}, \eta >_{L^2} + < Qw_t - Q_{n_k} w_{n_k,t}, \eta >_{L^2} \\ = \; & < w_x - w_{n_k,x}, \eta >_{L^2} + < w_t, (Q - Q_{n_k})\eta >_{L^2} + < w_t - w_{n_k,t}, Q_{n_k}\eta >_{L^2} . \end{aligned}$$

Here, $< \cdot, \cdot >_{L^2}$ denotes the standard inner product in $L^2(\mathbb{T})$. Since w_{n_k} weakly converges to w in $H^1(\mathbb{T})$ and $Q_{n_k}\eta$ is uniformly bounded by a square integrable function, the first and the third term converge to 0 as $n_k \to \infty$. The boundness of a weakly convergent sequence yields

$$\begin{aligned}
| < w_t, (Q - Q_{n_k})\eta >_{L^2} | &\leq ||w||_{H^1}||Q - Q_{n_k}||_{L^2}||\eta||_{H^1} \\
&\leq \liminf ||w_{n_k}||_{H^1}||Q - Q_{n_k}||_{L^2}||\eta||_{H^1} \\
&\leq \sqrt{2X}||w_0||_{H^1}||Q - Q_{n_k}||_{L^2}||\eta||_{H^1},
\end{aligned}$$

which goes to 0. Since w_{n_k} is a weak solution for Q_{n_k}, $< w_{n_k,x} + Q_{n_k} w_{n_k,t}, \eta >_{L^2} = 0$ for all n_k. Thus,

$$< w_x + Qw_t, \eta >_{L^2} = 0,$$

and this implies (5.44) as desired.

Since $H^1(\mathbb{T})$ is compactly embedded in $L^2(\mathbb{T})$,

$$w_{n_k} \to w \quad \text{in } L^2(\mathbb{T}). \tag{5.54}$$

Obviously, it also converges in $L^1(\mathbb{T})$, or

$$||w_{n_k}(x, \cdot) - w(x, \cdot)||_{L^1(x,2X-x)} \to 0 \quad \text{in } L^1(0, X). \tag{5.55}$$

Let

$$\mathbf{w}_{n_k}(x) = \int_x^{2X-x} |w_{n_k}(x, \cdot)|, \quad \mathbf{w}(x) = \int_x^{2X-x} |w(x, \cdot)|. \tag{5.56}$$

Then (5.55) yields

$$\mathbf{w}_{n_k}(x) \to \mathbf{w}(x) \quad \text{in } L^1(0, X).$$

This implies (5.52) almost all[2] x, since $|\mathbf{w}_{n_k}(x)| < ||w_0||_{L^1(0,2X)}$ for all n_k from (5.46).

Similarly, (5.47) gives

$$||w_{n_k}||_{L^1(\mathbb{T})} \leq X||w_0||_{L^1(0,2X)}.$$

Immediately, we obtain (5.53), since w_{n_k} converges to w in $L^1(\mathbb{T})$. □

Now, we prove the existence of direct problem, (5.43).

[2]Indeed, (5.52) holds for all x since $\mathbf{w}_{n_k}(x), \mathbf{w}(x)$ are continuous. To show the continuity, extend $w_{n_k}(x, \cdot), w(x, \cdot)$ so that they are in $L^2(I; H_0^1(J))$. Then one may show that they are in $C([0, X]; L^2(x, 2X - x))$ from Sobolev space theory (see §5.9 in [23]). Here, I, J are intervals containing $[0, X], [0, 2X]$ respectively.

Proof of existence in Proposition 5.4.3. Consider the following equation,

$$\widetilde{w}_x + Q(x)\widetilde{w}_t = 0 \quad \text{on } \mathbb{T}, \qquad \widetilde{w}(0,t) = \int_0^t w_0(s)ds, \quad 0 < t < 2X. \tag{5.57}$$

Since $w_0(t) \in L^2(0,2X)$, $\widetilde{w}(0,t) \in H^1(0,2X)$. Lemma 5.4.5 implies that there exists a weak solution $\widetilde{w} \in H^1(\mathbb{T})$ to (5.57). That is for any $\eta \in \mathcal{C}$,

$$\int_{\mathbb{T}} \tilde{w}^\dagger(\eta_{tx} + Q\eta_{tt}) + \int_0^{2X} \tilde{w}_0^\dagger \eta_t|_{x=0} dt = 0,$$

since $\eta_t \in \mathcal{C}$. Integration by parts yields

$$\int_{\mathbb{T}} \tilde{w}_t^\dagger(\eta_x + Q\eta_t) + \int_0^{2X} \tilde{w}_{0t}^\dagger \eta|_{x=0} dt = 0.$$

Let $w(x,t) = \widetilde{w}_t(x,t)$. Then $w(x,t) \in L^2(\mathbb{T})$ and $w(0,t) = w_0(t)$ almost everywhere. Hence a weak solution to (5.43) exists. □

The proof for the uniqueness of direct problem in Proposition 5.4.3 follows from an energy estimate described in Lemma 5.4.6.

Lemma 5.4.6. *A weak solution w to (5.43) satisfies the following equality.*

$$\int_{AD} |w|^2 dt = \int_{BC} |w|^2 dt + \int_{AB} |w|^2 - w^\dagger Q w dx + \int_{DC} |w|^2 + w^\dagger Q w dx. \tag{5.58}$$

Proof. Let $\mathbb{T}'$ be a subdomain of $\mathbb{T}$ shown in Figure 5.1. Since $C_c^\infty(\mathbb{T}')$ is a dense subset of $L^2(\mathbb{T}')$, there exists a sequence $\{w_n\}$ in $C_c^\infty(\mathbb{T}')$ that converges to w. For each n,

$$< w, w_{n,x} + Q w_{n,t} >= 0.$$

Now fix n. Since $w_n \rightharpoonup w$ and $w_{n,x} + Q w_{n,t} \in L^2(\mathbb{T}')$, for given $\varepsilon > 0$ there is M satisfying the following:

If $m > M$,

$$| < w_m, w_{n,x} + Q w_{n,t} > | < \varepsilon/2.$$

Thus,

$$| \int_{\mathbb{T}'} (w_m^\dagger w_n)_x + (w_m^\dagger Q w_n)_t | < \varepsilon.$$

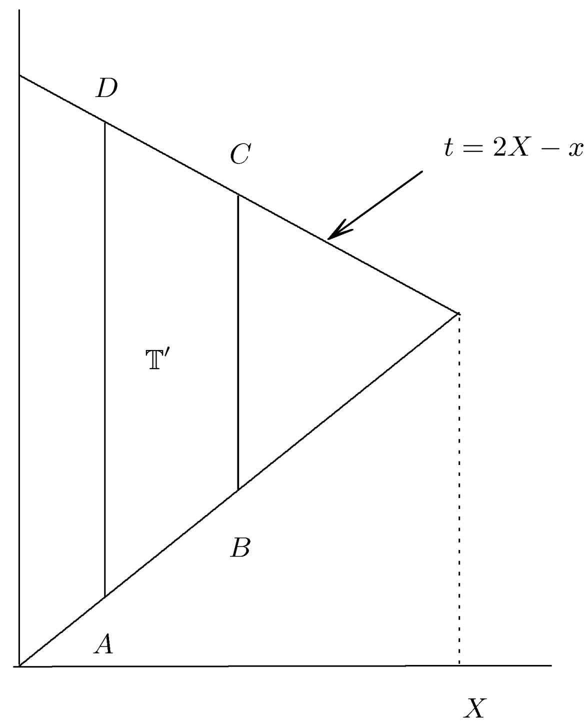

Figure 5.1 Domain $\mathbb{T}$ and $\mathbb{T}'$

Stokes' theorem over $\mathbb{T}'$ gives

$$|-\int_{AD} w_m^\dagger w_n dt + \int_{BC} w_m^\dagger w_n dt + \int_{AB} w_m^\dagger w_n - w_m^\dagger Q w_n dx + \int_{DC} w_m^\dagger w_n + w_m^\dagger Q w_n dx| < \varepsilon \quad (5.59)$$

(5.58) follows from a limit of (5.59), since $|w|^2 \geq |w^\dagger Q w|$. $\square$

Proof of uniqueness in Proposition 5.4.3. Let w_1, w_2 be solutions to (5.43). Define $v = w_1 - w_2$. Then v also solves (5.43) with zero initial condition. We apply (5.58) on an arbitrary trapezoid whose left vertical side AD is on the t-axis. Then

$$\int_{BC} |v|^2 dt + \int_{AB} |v|^2 - v^\dagger Q v dx + \int_{DC} |v|^2 + v^\dagger Q v dx = 0.$$

Since $|v|^2 \geq \pm v^\dagger Q v$, $v = 0$ almost everywhere on BC. BC is an arbitrary vertical line segment in $\mathbb{T}$, thus $v = 0$ a.e. in $\mathbb{T}$. $\square$

Due to the energy equality (5.58), we can show that weak solutions are continuously dependent on initial data as follows.

Corollary 5.4.7. *Suppose that $u, v \in L^2(\mathbb{T})$ are weak solutions to (5.43) with Q and initial data $u_0, v_0 \in L^2(0, 2X)$ respectively. Then*

$$||u - v||_{L^2(\mathbb{T})} \leq \sqrt{X}||u_0 - v_0||_{L^2(0,2X)}. \tag{5.60}$$

Proof. From the energy estimate for $u - v$,

$$\begin{aligned} ||u - v||^2_{L^2(\mathbb{T})} &= \int_0^X \int_x^{2X-x} |u - v|^2 dt dx \\ &\leq \int_0^X \int_0^{2X} |u_0 - v_0|^2 dt dx \\ &\leq X||u_0 - v_0||^2_{L^2(0,2X)}. \end{aligned}$$

□

5.4.2 Contraction Mapping

In the previous section, we showed the direct Time domain problem II is well defined for $Q \in L^2(0, X)$ and $\widehat{L} \in H^{-1}(0, 2X)$. For the inverse problem, we use a contraction mapping method. Recall the map $\mathfrak{L}$;

$$\begin{aligned} \mathfrak{L} : \overline{\mathcal{X}} &\rightarrow \overline{\mathcal{X}} \\ \mathbf{q} &\mapsto \begin{pmatrix} \dfrac{2u_1^* u_2}{|\mathbf{u}|^2} \\ \dfrac{|u_1|^2 - |u_2|^2}{|\mathbf{u}|^2} \end{pmatrix}, \end{aligned}$$

where,

$$\overline{\mathcal{X}} = \{ \begin{pmatrix} q_1 + q_2 i \\ q_3 \end{pmatrix} : q_i \in L^1(0, X), \quad |q_1|^2 + |q_2|^2 + |q_3|^2 = 1 \}.$$

q_1, q_2 and q_3 are components of a coefficient Q in the LLSP. $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the trace of $w(x, t)$ on the line segment $t = x$ for $0 < x < X$, where w a weak solution to (5.43) with Q given by $\mathbf{q}$, and the initial condition w_0 which is of bounded variation. Thus the map $\mathfrak{L}$ depends on w_0. Note that $\overline{\mathcal{X}}$ is a closed subset of $L^1(0, X)$. It is easy to check $|\mathfrak{L}(\mathbf{q})| = 1$ and $\mathfrak{L}(\mathbf{q})$ is integrable on $(0, X)$ provided the corresponding $|\mathbf{u}|$ is never zero on $(0, X)$. Thus we make the following hypothesis.

H: There are a closed subspace $\mathcal{X}$ of $\overline{\mathcal{X}}$ and $M > 0$ such that for any $\mathbf{q} \in \mathcal{X}$, the corresponding $\mathbf{u}$ satisfies $|\mathbf{u}| \geq M$ almost everywhere.

With this assumption H, we can show that the map $\mathfrak{L}$ is Lipschitz continuous in $\mathcal{X}$ as long as the initial data w_0 is of bounded variation.

For this aim, consider the following two problems on $\mathbb{T}$,

$$u_x + Qu_t = 0, \quad u(0,t) = w_0(t), \tag{5.61}$$

$$v_x + Pv_t = 0, \quad v(0,t) = w_0(t). \tag{5.62}$$

Here, Q, P are square integrable coefficients in the ZSSP. We assume that w_0 is in $H^2(0, 2X)$ for a moment. Then v_t is a solution of (5.62) with initial condition $w_{0,t}$. That is,

$$(v_t)_x + P(v_t)_x = 0, \quad (v_t)(0,t) = w_{0,t}.$$

Let $w = u - v$ and $f = (P - Q)v_t = Rv_t$. Then w solves

$$w_x + Qu_t - Pv_t = 0,$$

or

$$w_x + Qw_t = f, \quad w(0,t) = 0.$$

By Duhamel's principle, the solution is given by

$$w(x,t) = \int_0^x K(x - y, t, y)dy,$$

where, $K(z, t, y)$ solves

$$K_z + QK_t = 0 \quad \text{on } \mathbb{T}_y, \qquad K(0, t, y) = f(y, t).$$

Here, $\mathbb{T}_y$ is a subdomain of $\mathbb{T}$, see Figure 5.2. Note that $f(y, t) \in H^1(y, 2X - y)$ for almost all y, since we assume $w_0 \in H^2(0, 2X)$. From Lemma 5.4.5,

$$||K(\cdot, \cdot, y)||_{L^1(\mathbb{T}_y)} \leq (X - y)||f(y, \cdot)||_{L^1(y,2X-y)}. \tag{5.63}$$

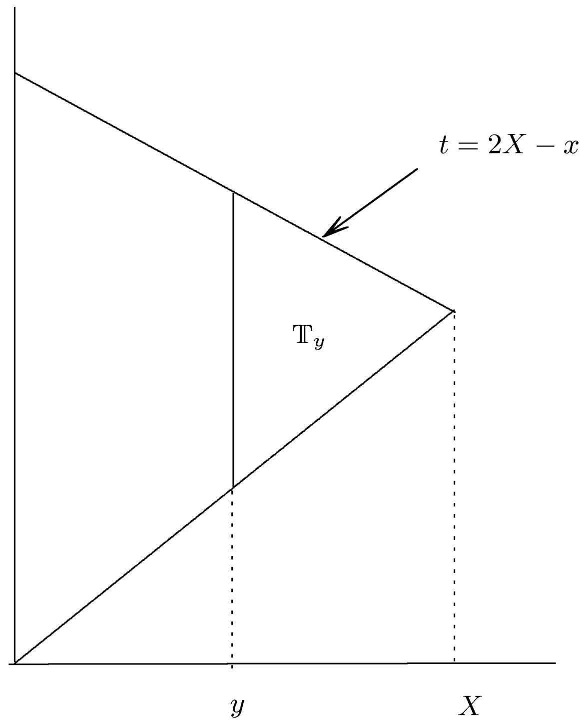

Figure 5.2 Domain $\mathbb{T}_y$

Now, we estimate $||w||_{L^1(\mathbb{T})}$.

$$
\begin{aligned}
\int_{\mathbb{T}} |w| &= \int_0^X \int_x^{2X-x} |\int_0^x K(x-y,t,y)dy|dtdx \\
&\le \int_0^X \int_y^X \int_x^{2X-x} |K(x-y,t,y)|dtdxdy \\
&\le \int_0^X \int_{DT_y} |K(x-y,t,y)|dtdxdy \\
&\le \int_0^X \int_y^{2X-y} (X-y)|f(y,t)|dtdy \quad \text{by } (5.63) \\
&= \int_0^X (X-y)|R(y)| \int_y^{2X-y} |v_t(y,t)|dtdy \\
&\le ||w_{0,t}||_{L^1(0,2X)} \int_0^X (X-y)|R(y)|dy \quad \text{by } (5.52) \\
&= \frac{1}{2}||w_{0,t}||_{L^1(0,2X)} \int_{\mathbb{T}} |R(x)|dtdx.
\end{aligned}
$$

We can apply this estimation to a triangle $\$ \subset \mathbb{T}$ whose right side is on the line $x = s$ and similar to $\mathbb{T}$.

$$\begin{aligned}\int_{\mathbb{S}}|w| &\leq \frac{1}{2}||w_t(s,\cdot)||_{L^1(s,2X-s)}\int_{\mathbb{S}}|R| \\ &\leq \frac{1}{2}||w_{0,t}||_{L^1(0,2X)}\int_{\mathbb{S}}|R|.\end{aligned}$$

Since $\mathbb{S}$ is an arbitrary triangle, for almost everywhere

$$|w| \leq \frac{1}{2}||w_{0,t}||_{L^1(0,2X)}|R|. \tag{5.64}$$

Let $\mathbf{p},\mathbf{q} \in \mathcal{X}$ be vectors corresponding P, Q in (5.61) and (5.62) respectively. Simple algebra gives

$$\begin{aligned}|\mathfrak{L}(\mathbf{p}) - \mathfrak{L}(\mathbf{q})|^2 &= \frac{4}{|\mathbf{u}|^2|\mathbf{v}|^2}|u_1v_2 - u_2v_1|^2 && (5.65)\\ &\leq 8\frac{|\mathbf{u}-\mathbf{v}|^2}{|\mathbf{v}|^2}. && (5.66)\end{aligned}$$

Together with (5.64), this yields

$$\begin{aligned}||\mathfrak{L}(\mathbf{p}) - \mathfrak{L}(\mathbf{q})||_{L^2(0,X)} &\leq \frac{\sqrt{2}}{\min|\mathbf{v}|}||w_{0,t}||_{L^1(0,2X)}||R||_{L^2(0,X)}, \\ &\leq \frac{\sqrt{2}}{M}||w_{0,t}||_{L^1(0,2X)}||\mathbf{p}-\mathbf{q}||_{L^2(0,X)}.\end{aligned}$$

Thus $\mathfrak{L}$ is Lipschitz continuous in $\mathcal{X}$ if w_0 is in $H^2(0,2X)$. This condition can be relaxed. Suppose that w_0 is of bounded variation. Then there a sequence $\{w_{0n}\} \subset W^{1,1}(0,2X)$ such that $w_{0n} \to w_0$ in $L^1(0,2X)$ and $||\dot{w}_{0n}||_{L^1(0,2X)} \to \mathrm{Var}(w_0)$ ([29]), where $\mathrm{Var}(w_0)$ is the total variation of w_0. Since the domain is bounded and w_0 is of bounded variation, we can find a subsequence $\{w_{0n_k}\} \in H^2(0,2X)$ such that $w_{0n_k} \to w_0$ in $L^2(0,2X)$ also. For notational simplicity, we use w_{0n} instead of w_{0n_k}. Note that $\mathfrak{L}$ is well defined for w_0 due to Proposition 5.4.3 and Lemma 5.4.6.

Let $\mathfrak{L}_n$ be the operator corresponding to $w_{0,n}$. Then for each $\mathbf{q} \in \mathcal{X}$,

$$\begin{aligned}||\mathfrak{L}(\mathbf{q}) - \mathfrak{L}_n(\mathbf{q})||_{L^2(0,X)} &\leq \frac{2\sqrt{2}}{M}||\mathbf{u}-\mathbf{u}_n||_{L^2(0,X)} \quad \text{by (5.66)}, \\ &\leq \frac{2\sqrt{2}}{M}||w_0 - w_{0n}||_{L^2(0,2X)} \quad \text{by Lemma 5.4.6}.\end{aligned}$$

Suppose that $\mathbf{p}, \mathbf{q} \in \mathcal{X}$. Then

$$\begin{aligned} ||\mathfrak{L}(\mathbf{p}) - \mathfrak{L}(\mathbf{q})||_{L^2} &\leq ||\mathfrak{L}_n(\mathbf{p}) - \mathfrak{L}_n(\mathbf{q})||_{L^2} + ||\mathfrak{L}(\mathbf{p}) - \mathfrak{L}_n(\mathbf{p})||_{L^2} + ||\mathfrak{L}(\mathbf{q}) - \mathfrak{L}_n(\mathbf{q})||_{L^2} \\ &\leq ||\mathfrak{L}_n(\mathbf{p}) - \mathfrak{L}_n(\mathbf{q})||_{L^2} + C||w_0 - w_{0n}||_{L^2(0,2X)} \\ &\leq \frac{\sqrt{2}}{M}||\dot{w}_{0n}||_{L^1(0,2X)}||\mathbf{p} - \mathbf{q}||_{L^2} + C||w_0 - w_{0n}||_{L^2(0,2X)}. \end{aligned}$$

It follows that

$$||\mathfrak{L}(\mathbf{p}) - \mathfrak{L}(\mathbf{q})||_{L^2} \leq \frac{\sqrt{2}}{M}\mathrm{Var}(w_0)||\mathbf{p} - \mathbf{q}||_{L^2}.$$

Hence, $\mathfrak{L}$ is well defined for w_0 of bounded variation, and Lipschitz continuous.

It is not hard to show that $\mathfrak{L}^n$ is a contraction mapping on $\mathcal{X}$ for sufficiently large n if $w_0 \in H^1$. However, if w_0 has jumps then we have to assume that $\sup_t |w(x,t^+) - w(x,t^-)|$ is small enough for each x to have the same result. We conjecture that

$$\sup_t |w(x,t^+) - w(x,t^-)| \leq C \sup_t |w_0(t^+) - w_0(t^-)|.$$

Although the theory about $\mathfrak{L}$ is incomplete, we have a numerical evidence that $\mathfrak{L}$ has a fixed point.

Figure 5.3 shows a reconstructed Q via the following iteration,

$$\mathbf{q}_{n+1} = \mathfrak{L}\mathbf{q}_n$$

Q is obtained after 7 iterations with initial guess $\mathbf{q}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ which is corresponding to $Q = \Lambda$. The residue is 6.8435×10^{-6}. We use the finite difference method to solve the direct problem.

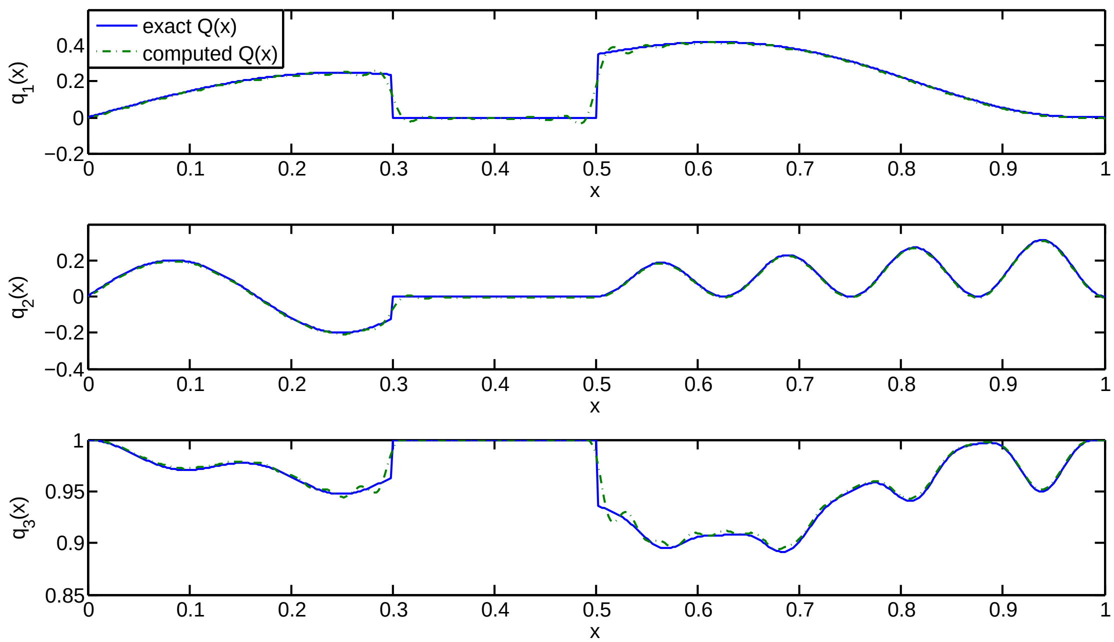

Figure 5.3 Reconstructed Q via the mapping $\mathfrak{L}$

CHAPTER 6. Further remarks and future works

1. *Existence of the ZSSP:*

   In Theorem 2.2.5, we defined a map from coefficients $s(x)$ of the ZSSP to left reflection coefficients $L(\zeta)$ and showed that it is injective from $\mathcal{S}_0$ to H_+^2. However, the range of the map is strictly subset of H_+^2. Thus the existence of the ZSSP fails in this case. One way to overcome this difficulty is to extend the domain. We might consider $s(x) \in L^2$ instead of $L^1 \cap L^2$. Although the scattering data may not defined in classical sense, we can approximate the scattering data of $s(x)$ by scattering data of $\{s_n(x)\}$ which converges to $s(x)$ in L^2 sense. For this aim, we need a Plancherel kind of identity between $s(x)$ and $L(\zeta)$. We have only one side inequality, for example, see [46]. It is unsolved yet if the L^2 norm of reflection coefficient is bonded by L^2 norm of coefficient s. One might define the domain of coefficients in a different space to have the existence. Note that Villarroel et al. identified function spaces in which the map is bijective ([52]).

2. *Modified Newton's method:*

   In Section 2.3, we introduced a modified Newton's method, and showed the iteration scheme (2.73) converges for a sufficiently small ω as long as Γ is bounded in some open ball $\mathcal{B}(s;\varepsilon)$ and the initial guess $s_0(x) \in \mathcal{B}(s;\varepsilon)$. We have to verify the properties of the map Γ to remove this assumption. In particular, the boundness of Γ is important and it might be related to the Plancherel identity which we discussed in previous part.

   This method may yield an ill-posed problem if $s(x)$ has bound states, thus the reconstructed $s(x)$ is less accurate than one from the Darboux transformation method. Nevertheless, this modified Newton's method is worth to be studied intensively, because it is more useful to solve the whole line problems. In Section 4.2, we discussed how half line reflection coefficients are extracted from $\{R(\zeta), \zeta_n, C_{l,n}\}_{n=1}^{N}$. Once $R_1(\zeta), L_2(\zeta)$ are given, we can apply the idea in Section 2.5 to find the bound state data. This procedure, however, is seriously ill posed problem as we mentioned in Section 2.6. The modified Newton's method might overcome this difficulty.

One can compute $\{R(\zeta), T(\zeta)\}$ from the standard scattering data via the spectral representation (1.21). Then $\mathcal{T}_2$, thus $\{L_2(\zeta), T_2(\zeta)\}$ can be extracted from the GLM equation as we discussed, and from (4.15),

$$T_1 = \frac{1}{a_1} = \frac{1}{a_2^* a + b_2 b^*} = \frac{T_2^* T}{1 + R_2^* R}.$$

We still have a numerical difficulty in computing $T(\zeta)$ and applying the modified Newton's method, but we expect these might cause weaker ill-posedness than the method we discussed in Section 4.2 in case of $N \geq 2$. Here, N is the number of bound states.

3. *Transformation on less regular coefficients:*

The regularity of the coefficients is decreasing under the transformation $\mathfrak{F}$ which is defined in Chapter 3. Thus, it might be useful to consider the inverse transformation, i.e. a transformation from the ZSSP to the LLSP or the ALLSP, in order to solve the ZSSP with less regular coefficient, e.g. δ- function. This coefficient would correspond to a discontinuous Q in the LLSP which might be solved by the mapping $\mathfrak{L}$. However, we have to define a product of distributions to justify the transformation $\mathfrak{F}$. Indeed, $\mathfrak{F}$ is governed by

$$F(x) = \int_0^x \frac{1}{2} Q_x(y) Q(y) F(y) + F_0.$$

If $Q(x)$ has a jump at $x = x_0$, then $Q_x(x)$ should be understood as a δ-function. Thus we have to define the kernel as a product of distributions. Some definitions of product of distributions were suggested, see e.g. [14], [24]. But it is not easy to clarify the transformation $\mathfrak{F}$ with their definitions.

4. *Anisotropic model of easy-plane:*

One of the motivation of the transformation $\mathfrak{F}$ is the equivalence of the cubic Schrödinger equation and the LL equation which was suggested by Lakshmanan ([39]). In [43], Mikeska showed the sine-Gordon equation is connected to the easy plain magnetic chain in a magnetic field. Since the sine-Gordon equation can be transformed to a Zakharov-Shabat kind scattering problem, it is natural to seek a relation of the ZSSP and the LLSP with anisotropy of the easy-plane type. According to [15], the corresponding scattering problem may have the following format;

$$\frac{\partial \psi}{\partial x} = i\zeta Q\psi + i\beta\Lambda\mathcal{D}^c Q\psi, \quad \beta \in \mathbb{R}. \tag{6.1}$$

But the reflection coefficients are defined only subset of $\mathbb{R}$. This incomplete data makes the problem challenging. If we assume that $R(\zeta)$ or $L(\zeta)$ is given on the whole real line, then the solution to (6.1) might be given by a Born approximation described in Section 2.6.1. For the exact solution, one can consider a transformation to the generalized ZSSP. The time domain approach and the contraction mapping idea might be adapted to solve the generalized ZSSP.

5. *Bound states:*

We mentioned that the bound state information is essential to solve a whole line scattering problem. Moreover, from the physical point of view, the bound states are connected to solitary wave solutions[1] to evolution equations. In the Schrödinger scattering problem, many nice properties about the bound states are known well, see e.g [18]. For example, the eigenvalues ζ_0 should be a pure imaginary number and it is a simple zero of $a(\zeta)$, the reciprocal of the transmission coefficient, in the upper half plane. Also it is known that $a(\zeta)$ never vanishes on the real line. That is, in the Schrödinger scattering problem we do not have to assume the generic condition (1.15). The number of bound states for a fragment of a potential can be estimated from the total number of bound states. More precisely, the total number of bound states N satisfies the following inequalities ([9]).

$$1 - p + \sum_{j=1}^{p} N_j \leq N \leq \sum_{j=1}^{p} N_j, \tag{6.2}$$

[1] The coefficient $s(x)$ given by (2.105) in turn is a soliton solution to the cubic Schrödinger equation.

where p is the number of fragments and N_j is the number of bound states for the jth fragment. The inequality might be useful to apply the splitting method ([10]).

On the other hand, the ZSSP has quite different properties. We have seen an example which has a non pure imaginary eigenvalue in Figure 3.1. Indeed, Klaus and Shaw proved that the eigenvalues are pure imaginary when $s(x)$ is a single lobe[2] in [36]. They also found the threshold energy for the existence of eigenvalues([37]); if

$$\int_{\mathbb{R}} |s(x)|dx \leq \frac{\pi}{2}, \tag{6.3}$$

then there are no eigenvalues. With aid of this, it is easy to check that the second inequality in (6.2) does not hold for the ZSSP. Nevertheless, we believe that there is an analogy to (6.2) in the ZSSP.

The threshold energy (6.3) can be easily adapted to the LLSP through the transformation $\mathfrak{F}$. Indeed,

$$Q_x^2 = 4F_xF_x^\dagger = 4|s|^2I.$$

Thus, we can state that there are no eigenvalues in the LLSP if

$$||Q_x||_2 \leq \pi.$$

The transformation $\mathfrak{F}$, however, does not give a clear picture how a single lobe coefficient $s(x)$ in the ZSSP is transformed to a coefficient $Q(x)$ of the LLSP.

6. *Generic condition:*

From the inequality (2.28), we expect that new eigenvalues may emerge or disappear at the real zero of $a(\zeta)$ if $s(x)$ is perturbed. Also it is known that the set of coefficients satisfying the condition (1.15) is a dense subset of general coefficients ([3]). However, the generic condition is not completely understood. Most of results in the ZSSP can not be obtained without assuming (1.15).

We remark that one can not obtain the uniqueness of the whole line problem from the splitting method directly. Even if $s(x)$ satisfies the generic condition, $a_j(\zeta)$ corresponding

[2] A coefficient $s(x)$ is called a single lobe if $s(x)$ is real valued L^1 function, bounded, piecewise smooth, and nondecreasing to the left of $x = 0$ and nonincreasing to the right of $x = 0$.

to a fragment $s_j(x)$ of $s(x)$ may vanish on the real line. That is, the split coefficients $s_1(x)$ or $s_2(x)$ may not in the classes $\mathcal{S}^p_{l,n}$ or $\mathcal{S}^p_{r,n}$ defined[3] in Chapter 3. Thus we may not apply the uniqueness theorem on the half line, Theorem 2.5.1, to obtain the uniqueness of the whole line problem. Instead, one can show that for a given coefficient $s(x) \in L^1$ satisfying the generic condition, there exists $x_0 \in \mathbb{R}$ such that $s\chi_{(-\infty,x_0)}$ and $s\chi_{(x_0,\infty)}$ satisfy the generic condition, from the inequality (2.28). Since two fragments $s\chi_{(-\infty,x_0)}$ and $s\chi_{(x_0,\infty)}$ can be understood as coefficients in $\mathcal{S}^p_{l,n}$ and $\mathcal{S}^p_{r,n}$ respectively, we have the uniqueness for the whole line problem.

7. *Time domain approach:*

   In Chapter 5, we suggested iteration method to solve the LLSP based on hyperbolic system of equations, although some mathematical proofs are incomplete. However, we believe that this time domain approach will give a uniqueness theorem for the LLSP in a certain class. Also, this idea may be adapted to the anisotropic model of Landau-Lifschitz scattering problem.

[3] Compare the definitions (3.6) and (4.3)

BIBLIOGRAPHY

[1] M. J. Ablowitz and P. A. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.

[2] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. Nonlinear-evolution equations of physical significance. *Phys. Rev. Lett.*, 31:125–127, 1973.

[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Appl. Math.*, 53(4):249–315, 1974.

[4] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. *Discrete and continuous nonlinear Schrödinger systems*, volume 302 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004.

[5] M. J. Ablowitz and H. Segur. *Solitons and the inverse scattering transform*, volume 4 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981.

[6] T. Aktosun. A factorization of the scattering matrix for the Schrödinger equation and for the wave equation in one dimension. *J. Math. Phys.*, 33(11):3865–3869, 1992.

[7] T. Aktosun. Bound states and inverse scattering for the Schrödinger equation in one dimension. *J. Math. Phys.*, 35(12):6231–6236, 1994.

[8] T. Aktosun. Inverse scattering transform, KdV, and solitons. In *Current trends in operator theory and its applications*, volume 149 of *Oper. Theory Adv. Appl.*, pages 1–22. Birkhäuser, Basel, 2004.

[9] T. Aktosun, M. Klaus, and C. van der Mee. On the number of bound states for the one-dimensional Schrödinger equation. *J. Math. Phys.*, 39(9):4249–4256, 1998.

[10] T. Aktosun and P. E. Sacks. Potential splitting and numerical solution of the inverse scattering problem on the line. *Math. Methods Appl. Sci.*, 25(4):347–355, 2002.

[11] E. Barouch, A. S. Fokas, and V. G. Papageorgiou. The bi-Hamiltonian formulation of the Landau-Lifshiftz equation. *J. Math. Phys.*, 29(12):2628–2633, 1988.

[12] R. Beals and R. R. Coifman. Scattering and inverse scattering for first order systems. *Comm. Pure Appl. Math.*, 37(1):39–90, 1984.

[13] R. Beals and R. R. Coifman. Inverse scattering and evolution equations. *Comm. Pure Appl. Math.*, 38(1):29–42, 1985.

[14] L. Berg. Representations for distribution algebras. *Z. Angew. Math. Mech.*, 56(5):177–181, 1976.

[15] A. E. Borovik, S. Klama, and S. I. Kulinich. Integration of the Landau-Lifshitz equation with preferred-axis anisotropy by the method of the inverse scattering problem. *Phys. D*, 32(1):107–134, 1988.

[16] K. P. Bube and R. Burridge. The one-dimensional inverse problem of reflection seismology. *SIAM Rev.*, 25(4):497–559, 1983.

[17] F. Calogero and A. Degasperis. *Spectral transform and solitons. Vol. I*, volume 13 of *Studies in Mathematics and its Applications.* North-Holland Publishing Co., Amsterdam, 1982. Tools to solve and investigate nonlinear evolution equations, Lecture Notes in Computer Science, 144.

[18] K. Chadan and P. C. Sabatier. *Inverse problems in quantum scattering theory.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1989. With a foreword by R. G. Newton.

[19] A. Cohen and T. Kappeler. Scattering and inverse scattering for steplike potentials in the Schrödinger equation. *Indiana Univ. Math. J.*, 34(1):127–180, 1985.

[20] A. Cohen and T. Kappeler. Solutions to the cubic Schrödinger equation by the inverse scattering method. *SIAM J. Math. Anal.*, 23(4):900–922, 1992.

[21] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[22] P. Deift and E. Trubowitz. Inverse scattering on the line. *Comm. Pure Appl. Math.*, 32(2):121–251, 1979.

[23] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 1998.

[24] B. Fisher, E. Özçaḡ, and Ü. Gülen. A theorem on the commutative neutrix product of distributions. *Sarajevo J. Math.*, 1(14)(2):235–242, 2005.

[25] H. C. Fogedby. Solitons and magnons in the classical Heisenberg chain. *J. Phys. A*, 13(4):1467–1499, 1980.

[26] G. Freiling and V. Yurko. *Inverse Sturm-Liouville problems and their applications.* Nova Science Publishers Inc., Huntington, NY, 2001.

[27] P. R. Garabedian. *Partial differential equations.* AMS Chelsea Publishing, Providence, RI, 1998. Reprint of the 1964 original.

[28] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura. Method for solving the korteweg-devries equation. *Phys. Rev. Lett.*, 19(19):1095–1097, Nov 1967.

[29] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[30] C. Gu, H. Hu, and Z. Zhou. *Darboux transformations in integrable systems*, volume 26 of *Mathematical Physics Studies*. Springer, Dordrecht, 2005. Theory and their applications to geometry.

[31] J. K. Hale. *Ordinary differential equations*. Wiley-Interscience [John Wiley & Sons], New York, 1969. Pure and Applied Mathematics, Vol. XXI.

[32] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[33] N.-N. Huang, H. Cai, T. Yan, and F.-R. Xu. Hamiltonian formalism of the Landau-Lifschitz equation for a spin chain with full anisotropy. *J. Nonlinear Math. Phys.*, 13(2):302–314, 2006.

[34] F. John. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition, 1982.

[35] M. Klaus. On the Zakharov-Shabat eigenvalue problem. In *Mathematical studies in nonlinear wave propagation*, volume 379 of *Contemp. Math.*, pages 21–45. Amer. Math. Soc., Providence, RI, 2005.

[36] M. Klaus and J. K. Shaw. Purely imaginary eigenvalues of zakharov-shabat systems. *Phys. Rev. E*, 65(3):036607, Feb 2002.

[37] M. Klaus and J. K. Shaw. On the eigenvalues of Zakharov-Shabat systems. *SIAM J. Math. Anal.*, 34(4):759–773 (electronic), 2003.

[38] R. Kress. *Linear integral equations*, volume 82 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1989.

[39] M. Lakshmanan. Continuum spin system as an exactly solvable dynamical system. *Phys. Lett. A*, 61:53–54, Apr. 1977.

[40] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[41] J. M. Lin. Evolution of the scattering data under the classical Darboux transform for su(2) soliton systems. *Acta Math. Appl. Sinica (English Ser.)*, 6(4):308–316, 1990.

[42] J. H. Liu. *A first course in the qualitative theory of differential equations.* Prentice Hall Inc., Upper Saddle River, NJ, 2003.

[43] H. J. Mikeska. Solitons in a one-dimensional magnet with an easy plane . *Journal of Physics C Solid State Physics*, 11:L29–L32, Jan. 1977.

[44] N. N. Novikova and V. M. Markushevich. Uniqueness of the solution of the one-dimensional problem of scattering for potentials located on the positive semiaxis. *Comput. Seismology*, 18:176–184, 1987.

[45] Rakesh. A one-dimensional inverse problem for a hyperbolic system with complex coefficients. *Inverse Problems*, 17(5):1401–1417, 2001.

[46] P. Sacks. An inverse problem in coupled mode theory. *J. Math. Phys.*, 45(4):1699–1710, 2004.

[47] P. E. Sacks. Reconstruction of steplike potentials. *Wave Motion*, 18(1):21–30, 1993.

[48] F. Santosa and H. Schwetlick. The inversion of acoustical impedance profile by methods of characteristics. *Wave Motion*, 4(1):99–110, 1982.

[49] J. K. Shaw. *Mathematical principles of optical fiber communications*, volume 76 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2004.

[50] E. K. Sklyanin. On complete integrability of the landau-lifshitz equation. *Steklov Mathematical Institute Leningrad Branch preprint LOMI*, E-3-79, 1979.

[51] L. A. Takhtajan. Integration of the continuous Heisenberg spin chain through the inverse scattering method. *Phys. Lett. A*, 64(2):235–237, 1977.

[52] J. Villarroel, M. J. Ablowitz, and B. Prinari. Solvability of the direct and inverse problems for the nonlinear Schrödinger equation. *Acta Appl. Math.*, 87(1-3):245–280, 2005.

[53] L. Weiss and R. N. McDonough. Prony's method, Z-transforms, and Padé approximation. *SIAM Rev.*, 5:145–149, 1963.

[54] V. E. Zaharov and L. A. Tahtadžjan. Equivalence of a nonlinear Schrödinger equation and a Heisenberg ferromagnet equation. *Teoret. Mat. Fiz.*, 38(1):26–35, 1979.

[55] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.

[56] X. Zhou. Zakharov-shabat inverse scattering. *in Scattering: Scattering and Inverse Scattering in Pure and Applied Science, edited by R. Pike and P. Sabatier (Academic, New York)*, (1):1707–1716, 2002.

ACKNOWLEDGEMENTS

I would like to deeply thank my advisor, Prof. Sacks for his tremendous help and expert advice, and Mr. Kim, Keum Young who first taught me mathematics.